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ALGEBRAIC GAMMA MONOMIALS AND DOUBLE COVERINGS OF CYCLOTOMIC FIELDS

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Abstract. We investigate the properties of algebraic gamma monomials—that is, algebraic numbers which are expressible as monomials in special values of the classical gamma function. Recently Anderson has constructed a double complex $\mathbb{SK}$, to compute $H^*(\pm, \mathbb{U})$, where $\mathbb{U}$ is the universal ordinary distribution. We use the double complex to deduce explicit formulae for algebraic gamma monomials. We provide simple proofs of some previously known results of Deligne on algebraic gamma monomials. Deligne used the theory of Hodge cycles for his results. By contrast, our proofs are constructive and relatively elementary. Given a Galois extension K/F, we define a *double covering* of K/F to be an extension $\tilde{K}/K$ of degree ≤ 2, such that $\tilde{K}/F$ is Galois. We demonstrate that each class $\mathbf{a} \in H^2(\pm, \mathbb{U})$ gives rise to a double covering of $\mathbb{Q}(\zeta_\infty)/\mathbb{Q}$, by $\mathbb{Q}(\zeta_\infty, \sqrt{\sin \mathbf{a}})/\mathbb{Q}(\zeta_\infty)$. When $\mathbf{a}$ lifts a canonical basis element indexed by two odd primes, we show that this double covering can be non-abelian. *However, if* $\mathbf{a}$ *represents any of the canonical basis classes indexed by an odd squarefree positive integer divisible by at least four primes, then the Galois group of* $\mathbb{Q}(\zeta_\infty, \sqrt{\sin \mathbf{a}})/\mathbb{Q}$ *is abelian and hence* $\sqrt{\sin \mathbf{a}} \in \mathbb{Q}(\zeta_\infty)$. The $\sqrt{\sin \mathbf{a}}$ may very well be a new supply of abelian units. The relevance of these units to the unit index formula for cyclotomic fields calls for further investigations.

1. Introduction

In this paper we investigate the properties of algebraic gamma monomials—that is, algebraic numbers which are expressible as monomials in special values of the classical gamma function. The classical gamma function satisfies the following well-known functional equations:

$$\Gamma(s+1) = s\Gamma(s), \tag{1}$$

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \tag{2}$$

$$(2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2}-s}\Gamma(s) = \prod_{i=0}^{n-1} \Gamma(\frac{s+i}{n}). \tag{3}$$

These functional equations can be used to show that various products of special values of the gamma function are algebraic. For example, (3) tells us that the

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gamma monomial

$$\frac{\Gamma(2/15)\Gamma(5/15)\Gamma(8/15)\Gamma(11/15)\Gamma(14/15)}{\Gamma(2/3)\Gamma(1/2)^4} = 4 \cdot 5^{-1/6}$$

is algebraic. However, algebraic gamma monomials also arise in more subtle ways—that is, not as " $\mathbb{Z}$-linear combinations of (1), (2) and (3)". For example,

$$\frac{\Gamma(1/3)\Gamma(2/15)}{\Gamma(4/15)\Gamma(1/5)} = 3^{1/5}5^{-1/12}\sqrt{\frac{\sin(4\pi/15)\sin(\pi/5)}{\sin(\pi/3)\sin(2\pi/15)}}.$$

In fact the *square* of the above relation is a $\mathbb{Z}$-linear combination of (1), (2) and (3). In the sequel we will describe a method for generating such monomial relations. First we give a quick tour of the research that has been conducted in this area. Details will appear later in the sequel.

Koblitz - Ogus. Let $\mathbb{A}$ be the free abelian group generated by symbols of the form $[a]$, where $a \in \mathbb{Q}/\mathbb{Z}$. Let $\mathbf{a} = \sum m_i\,[a_i] \in \mathbb{A}$. We write

$$\Gamma(\mathbf{a}) = \prod_{i:\ a_i \neq 0} \left(\frac{\sqrt{2\pi}}{\Gamma(\langle a_i \rangle)} \right)^{m_i}.$$

Here $\langle a_i \rangle$ is the smallest positive rational number representing the class of a_i in $\mathbb{Q}/\mathbb{Z}$. Koblitz and Ogus [4] have shown that $\Gamma(\mathbf{a})$ is algebraic if $\sum m_i \langle a_i \rangle = \sum m_i \langle t a_i \rangle$ for all $t \in (\mathbb{Z}/f\mathbb{Z})^\times$, where f is the lcm of the denominators of the $\langle a_i \rangle$ (the $\langle a_i \rangle$ are in their lowest terms). The converse of this is a major unsolved conjecture (Rohrlich's conjecture) in transcendental number theory.

Sinnott, Kubert. Building on Sinnott's ideas from [8], Kubert [5, 6] computed $H^*(\pm, \mathbb{U})$, where $\mathbb{U}$ is the universal ordinary distribution (see Section 2 for the definition). Kubert showed that $H^2(\pm, \mathbb{U})$ can be identified with the torsion subgroup of $\mathbb{U}^-$, where $\mathbb{U}^-$ is the universal odd distribution. It follows from Kubert's result that $\mathbf{a} \in H^2(\pm, \mathbb{U})$ if and only if $\mathbf{a}$ satisfies the Koblitz and Ogus internal sum criterion given above. In particular, if $\mathbf{a} \in H^2(\pm, \mathbb{U})$, then $\Gamma(\mathbf{a})$ is algebraic.

Deligne (connections with algebraic and arithmetic geometry). Any $\mathbf{a} \in H^2(\pm, \mathbb{U})$ which is "effective and integral" gives rise to a Hodge cycle on a Fermat hypersurface. Deligne [2, 3] showed that the Hodge cycle related to $\mathbf{a}$ is "absolutely Hodge". This led to the proof of Deligne's reciprocity law which relates algebraic gamma monomials and Jacobi sum Hecke characters.

Anderson's double complex. Recently Anderson [1] has constructed a double complex $\mathbb{SK}$ such that H_0 and H_1 of the total complex $Tot(\mathbb{SK})$ can be identified with $H^2(\pm, \mathbb{U})$ and $H^1(\pm, \mathbb{U})$ respectively. $\mathbb{SK}$ is generated as a free abelian group by symbols of the form $[a, g, n]$, where $a \in \mathbb{Q}/\mathbb{Z}$, g is a square-free, positive integer, and n is any integer. The free abelian group $\mathbb{SK}$ is bigraded by declaring the generator $[a, g, n]$ to be of bidegree (m, n), where m is the number of prime factors of g. Using the double complex, it is possible to define a canonical $\mathbb{F}_2$-basis for $H^*(\pm, \mathbb{U})$. In fact, the canonical $\mathbb{F}_2$-basis for $H^2(\pm, \mathbb{U})$ is indexed by squarefree positive integers with an even number of prime divisors.

Summary of results. We begin by deducing certain homological results concerning the structure of the double complex. We prescribe a "canonical lifting" procedure for lifting generators to cycles representing non-trivial elements in $H^2(\pm, \mathbb{U})$.

Using canonical lifting, we deduce explicit formulae for the algebraic gamma monomials under consideration. Canonical lifting makes it possible, for example, to write Maple codes for performing computations with algebraic gamma monomials.

We define $\sin \mathbf{a} = \prod_{i:a_i \not\equiv 0} (2 \sin \pi \langle a_i \rangle)^{m_i}$, where $\mathbf{a} = \sum m_i \,[a_i] \in H^2(\pm, \mathbb{U})$, with notation as earlier. We show that $(\Gamma(\mathbf{a}))^{2f}/(\sin \mathbf{a})^f$ is the square root of a rational number, where f is the lcm of the denominators of the $\langle a_i \rangle$. We assume that f is odd for simplicity. (It should not be difficult to extend our results to the case when $2 \mid f$). Using the Koblitz-Ogus criterion, we show that the factorization of this rational must possess an even number of primes congruent to $3 \pmod 4$. This gives a simple proof (without the use of Hodge cycles) of the fact (due to Deligne [2]) that $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a}))$ is a Kummer extension of $\mathbb{Q}(\zeta_f)$.

We deduce a necessary criterion for an element of $\mathbb{A}$ to be in $H^1(\pm, \mathbb{U})$, similar in spirit to the Koblitz-Ogus criterion for elements in $H^2(\pm, \mathbb{U})$. Using this, we are able to provide a simple proof of the fact (Deligne [2]) that $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a}))/\mathbb{Q}$ is a Galois extension, and depends only on the class of $\mathbf{a}$ in $H^2(\pm, \mathbb{U})$. When $\mathbf{a}$ lifts a canonical basis element indexed by two odd primes, this proof involves Gauss' lemma, which appears in the proof of the quadratic reciprocity law. When $\mathbf{a}$ lifts a canonical basis element generated by four or more odd primes, the proof relies on our necessary criterion for $H^1(\pm, \mathbb{U})$.

Given a Galois extension K/F, we define a *double covering* of K/F to be an extension $\tilde{K}/K$ of degree ≤ 2, such that $\tilde{K}/F$ is Galois. Thus we demonstrate that each class in $H^2(\pm, \mathbb{U})$ gives rise to a double covering of $\mathbb{Q}(\zeta_\infty)/\mathbb{Q}$, by $\mathbb{Q}(\zeta_\infty, \sqrt{\sin \mathbf{a}})/\mathbb{Q}(\zeta_\infty)$. When $\mathbf{a}$ lifts a canonical basis element indexed by two odd primes, we show that this double covering can be non-abelian. *However, if* $\mathbf{a}$ *represents any of the canonical basis classes indexed by an odd squarefree positive integer divisible by at least four primes, then the Galois group of* $\mathbb{Q}(\zeta_\infty, \sqrt{\sin \mathbf{a}})/\mathbb{Q}$ *is abelian and hence* $\sqrt{\sin \mathbf{a}} \in \mathbb{Q}(\zeta_\infty)$. The proof of this uses mostly the techniques and results described above. However, there is a crucial step which involves Deligne's reciprocity and hence relies on the theory of absolute Hodge cycles.

2. The universal ordinary distribution

We list the main features of ordinary distributions used in the sequel. For a detailed discussion of these and other results, we refer to the papers by Kubert [5, 6], and the texts by Lang [7] and Washington [9, chapter 12]. First, we make the following definitions:

Definition 1. Let $\mathbb{A}$ be the free abelian group generated by symbols of the form $[a]$, where $a \in \mathbb{Q}/\mathbb{Z}$. Let $\mathbb{U}$ be the quotient of $\mathbb{A}$ by the subgroup generated by all elements of $\mathbb{A}$ of the form $[a] - \sum_{nb=a} [b]$, where n is a positive integer and $a \in \mathbb{Q}/\mathbb{Z}$. Define $u\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{U}$ to be the map induced by $a \mapsto [a] : \mathbb{Q}/\mathbb{Z} \to \mathbb{A}$. Then u is the *universal ordinary distribution*. By abuse of language, the group $\mathbb{U}$ itself is also referred to as the universal ordinary distribution.

Definition 2. Let $\mathbb{A}$ be as above. Define $\mathbb{U}^-$ to be the quotient of $\mathbb{A}$ by the subgroup generated by all elements of $\mathbb{A}$ of the form $[a] - \sum_{nb=a} [b]$, along with all those of the form $[a] + [-a]$, where n is a positive integer and $a \in \mathbb{Q}/\mathbb{Z}$. Define $u^-\colon \mathbb{Q}/\mathbb{Z} \to \mathbb{U}^-$ to be the map induced by $a \mapsto [a] : \mathbb{Q}/\mathbb{Z} \to \mathbb{A}$. Then u^- is the *universal odd distribution*. By abuse of language, the group $\mathbb{U}^-$ itself is also referred to as the universal odd distribution.

Similarly we define the *universal even distribution* $\mathbb{U}^+$ to be the quotient of $\mathbb{A}$ by the subgroup generated by all elements of $\mathbb{A}$ of the form $[a] - \sum_{nb=a} [b]$, along with all those of the form $[a] - [-a]$, where n is a positive integer and $a \in \mathbb{Q}/\mathbb{Z}$.

We note that the two-element group $\{\pm 1\}$ acts naturally on $\mathbb{U}$, since the involution $[a] \mapsto [-a] : \mathbb{A} \to \mathbb{A}$ descends to $\mathbb{U}$. Building upon the work of Sinnott [8], Kubert [5, 6] computed the cohomology groups $H^*(\{\pm 1\}, \mathbb{U})$. In the sequel we will use $\pm$ to denote the two-element group and write the cohomology groups described above as $H^*(\pm, \mathbb{U})$. Kubert proved the following propositions:

Proposition 1. *The universal ordinary distribution* $\mathbb{U}$ *is a free abelian group.*

Proposition 2. *One has the following canonical isomorphisms:*

$$(\mathit{Torsion})\mathbb{U}^- = H^2(\pm, \mathbb{U}) \quad \mathit{and} \quad (\mathit{Torsion})\mathbb{U}^+ = H^1(\pm, \mathbb{U}).$$

Remark. Let $c \colon \mathbb{U} \to \mathbb{U}$ be the involution of $\mathbb{U}$ induced by the involution $[a] \mapsto [-a] : \mathbb{A} \to \mathbb{A}$. Then, it can be shown that the first canonical isomorphism in Proposition 2 is explicitly given by

$$H^2(\pm, \mathbb{U}) = \frac{\mathit{ker}(1 - c \colon \mathbb{U} \to \mathbb{U})}{\mathit{image}(1 + c \colon \mathbb{U} \to \mathbb{U})} \subseteq \frac{\mathbb{U}}{\mathit{image}(1 + c \colon \mathbb{U} \to \mathbb{U})} = \mathbb{U}^-.$$

Similar comments also apply to $\mathbb{U}^+$.

In the sequel we will often abuse notation and write

$$\mathbf{a} = \sum m_i \, [a_i] \in H^2(\pm, \mathbb{U}).$$

By this we precisely mean that $\mathbf{a}$ is first identified as an element of $\mathbb{U}$, by taking its image modulo the subgroup generated by all elements of $\mathbb{A}$ of the form $[a] - \sum_{nb=a} [b]$, along with all those of the form $[a] + [-a]$. This image of $\mathbf{a}$ in $\mathbb{U}$ is then identified with an element of $H^2(\pm, \mathbb{U})$, with the help of the canonical isomorphism described above. That is, the image of $\mathbf{a}$ in $\mathbb{U}$, modulo $\mathit{image}(1 + c \colon \mathbb{U} \to \mathbb{U})$, lies in $\mathit{ker}(1 - c \colon \mathbb{U} \to \mathbb{U})$. Similarly we will often abuse notation and write $\mathbf{b} = \sum n_i \, [b_i] \in H^1(\pm, \mathbb{U})$. Finally, if $\mathbf{a} = \sum m_i \, [a_i] \in H^2(\pm, \mathbb{U})$, we may assume, for the questions considered in this paper, that no a_i is 0; precisely, if $\mathbf{a}'$ is formed from $\mathbf{a}$ by removing the terms for which $[a_i] = [0]$, then $\mathbf{a}'$ also lies in $H^2(\pm, \mathbb{U})$ and $\Gamma(\mathbf{a}) = \Gamma(\mathbf{a}')$.

3. The double complex $\mathbb{SK}$

The results in this section are due to Anderson. We describe the main features of the double complex that are used in the sequel. For more details we refer to Anderson [1].

Let $\mathbb{SK}$ be the free abelian group generated by symbols of the form

$$[a, g, j]\,, \text{ where } a \in \mathbb{Q}/\mathbb{Z}, \text{ and } g \text{ and } j \text{ are integers, with } g \text{ positive and squarefree.} \tag{4}$$

We bigrade the free abelian group $\mathbb{SK}$ by requiring $[a, g, j]$ to belong to the bidegree (i, j) component $\mathbb{SK}_{i,j}$ of $\mathbb{SK}$, where i is the number of prime factors of g.

We also let Λ be the polynomial ring over $\mathbb{Z}$ in independent variables X_p indexed by the prime numbers p. For all positive integers n, we define $X_n \in \Lambda$ by

$$X_n = \prod_i X_{p_i}^{e_i}, \quad \text{where} \quad n = \prod_i p_i^{e_i}$$

is the prime factorisation of n. Note that the collection

$$\{X_n\colon n \text{ a positive integer}\}$$

forms a basis for Λ as a free abelian group.

We prescribe a Λ-module structure on $\mathbb{SK}$, by defining

$$X_n\,[a,g,j] = \sum_{nb=a} [b,g,j]\,,$$

for all positive integers n, where $[a,g,j]$ runs through the basis (4). It can be shown that $\mathbb{SK}_{i,j}$ is Λ-stable and free as a Λ-module.

We define a Λ-linear differential δ of bidegree $(0,-1)$ by

$$\delta\,[a,g,j] = (-1)^i([a,g,j-1] - (-1)^j\,[-a,g,j-1]), \tag{5}$$

where $[a,g,j]$ runs through the basis (4), and (i,j) is the bidegree of $[a,g,j]$.

We also define a Λ-linear differential ∂ of bidegree $(-1,0)$ by

$$\partial\,[a,g,j] = \sum_{p|g} \epsilon(g,p)(1-X_p)\,[a,g/p,j]\,, \tag{6}$$

where $[a,g,j]$ runs through the basis (4), and the sum is over all prime divisors p of g. The term $\epsilon(g,p)$ is defined as follows. If the prime factorisation of g is

$$g = p_1 \dots p_m, \quad \text{with } p_1 < \dots < p_m,$$

then we define

$$\epsilon(g,p) = (-1)^i, \quad \text{where } p = p_i \text{ for some } i.$$

It can be shown from these definitions that $\delta^2 = 0$ and $\partial^2 = 0$. Furthermore

$$\partial\delta = -\delta\partial, \quad \text{so that} \quad (\partial+\delta)^2 = 0.$$

We let $Tot(\mathbb{SK})$ denote the total complex of $\mathbb{SK}$, with the differential

$$(\partial+\delta)\colon Tot_n(\mathbb{SK}) \to Tot_{n-1}(\mathbb{SK}),$$

of total degree -1. We will also write $(\mathbb{SK}, \partial+\delta)$ to denote the total complex of $\mathbb{SK}$ with the differential described above.

We now define $\mathbb{NSK}$ to be the subgroup of $\mathbb{SK}$ generated by $\beta(a,j)\,[a,g,j]$, where $[a,g,j]$ runs through the basis (4), and

$$\begin{aligned} \beta(a,j) &= 2 \quad \text{if } 2a=0 \text{ and } j \text{ is even,} \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

It can be shown that $\mathbb{NSK}$ is both δ-stable and ∂-stable, and so $(\partial+\delta)$-stable, i.e. $\partial\mathbb{NSK} + \delta\mathbb{NSK} \subseteq \mathbb{NSK}$. Furthermore, it can be shown that $\mathbb{NSK}$ is $(\partial+\delta)$-acyclic. With notation as described above, Anderson proves the following theorems, which we will use in the sequel.

First we let $H_0(\mathbb{SK}, \partial)$ denote the edge complex derived from $\mathbb{SK}$—that is, the graded abelian group whose term in degree n is the cokernel of $\partial\colon \mathbb{SK}_{1,n} \to \mathbb{SK}_{0,n}$, equipped with the differential induced by δ. Then $H_0(\mathbb{SK}, \partial)$ is canonically isomorphic to the complex

$$\dots \overset{1-c}{\longleftarrow} \underset{\text{degree } 0}{\mathbb{U}} \overset{1+c}{\longleftarrow} \underset{\text{degree } 1}{\mathbb{U}} \overset{1-c}{\longleftarrow} \dots,$$

where $c\colon \mathbb{U} \to \mathbb{U}$ is the involution of $\mathbb{U}$ as discussed earlier. Thus $H_0(H_0(\mathbb{SK}, \partial), \delta)$ is canonically isomorphic to the torsion subgroup of $\mathbb{U}^-$. Using this, Anderson derives the following isomorphisms:

Theorem 1. *There exist canonical isomorphisms*

$$\begin{aligned} H^2(\pm, \mathbb{U}) &= H_0(H_0(\mathbb{SK}, \partial), \delta) \\ &= H_0(\mathbb{SK}, \partial + \delta) = H_0(\mathbb{SK}/\mathbb{NSK}, \partial + \delta). \end{aligned}$$

Similarly, for $H^1(\pm, \mathbb{U})$ we have the following:

Theorem 2. *There exist canonical isomorphisms*

$$\begin{aligned} H^1(\pm, \mathbb{U}) &= H_{-1}(H_0(\mathbb{SK}, \partial), \delta) \\ &= H_{-1}(\mathbb{SK}, \partial + \delta) = H_{-1}(\mathbb{SK}/\mathbb{NSK}, \partial + \delta). \end{aligned}$$

In fact Anderson showed that for, all positive integers i and i' such that $i' \equiv i \bmod 2$, we have canonical isomorphisms $H^i(\pm, \mathbb{U}) = H_{i'}(\mathbb{SK}, \partial + \delta)$, and therefore $H^i(\pm, \mathbb{U}) = H_{i'}(\mathbb{SK}/\mathbb{NSK}, \partial + \delta)$. Finally, the canonical $\mathbb{F}_2$-basis for $H^*(\pm, \mathbb{U})$ is constructed as follows:

Theorem 3. *Given a squarefree positive integer g with i prime factors, we define $k_g \in \mathbb{SK}_{i,-i}/\mathbb{NSK}_{i,-i}$ by*

$$\begin{aligned} k_g &= [0, g, -i]\,, \quad \text{if } g \text{ is odd and } i \text{ is even}, \\ &= [0, g, -i-1]\,, \quad \text{if } g \text{ is odd and } i \text{ is odd}, \\ &= [0, g, -i] + [1/2, g, -i]\,, \quad \text{if } g \text{ is even and } i \text{ is even}, \\ &= [0, g, -i-1] + [1/2, g, -i-1]\,, \quad \text{if } g \text{ is even and } i \text{ is odd}. \end{aligned}$$

Then the collection $\{k_g\}$, for g ranging over squarefree positive integers with an even number of prime factors, forms an $\mathbb{F}_2$-basis for $H_0(\mathbb{SK}/\mathbb{NSK}, \partial + \delta)$. On the other hand, the collection $\{k_g\}$, for g ranging over squarefree positive integers with an odd number of prime factors, forms an $\mathbb{F}_2$-basis for $H_{-1}(\mathbb{SK}/\mathbb{NSK}, \partial + \delta)$.

Remark A. Theorem 3, along with the isomorphisms from Theorems 1 and 2, makes the double complex a very useful tool. For computational purposes, these results imply that cycles in $\mathbb{SK}$ lift classes belonging to the torsion subgroup of $\mathbb{U}^{\pm}$. This is illustrated in Section 9, where, given distinct odd primes p and q, with $p < q$, we lift the generator $k_{pq} = [0, pq, -2] \in \mathbb{SK}_{2,-2}/\mathbb{NSK}_{2,-2}$ to a $(\partial + \delta)$-cycle in $\mathbb{SK}$. From the bidegree $(0,0)$-component of the lifted cycle, we read off the element $\mathbf{a}_{pq}$ belonging to the torsion subgroup of $\mathbb{U}^-$.

Remark B. Given a positive integer f, let $\mathbb{SK}^{(f)}$ be the subgroup of $\mathbb{SK}$ generated by all symbols of the form $[a, g, j]$, where $a \in \frac{g}{f}\mathbb{Z}/\mathbb{Z}$, j is any integer, and g is a squarefree positive integer dividing f. Also let $\mathbb{NSK}^{(f)} = \mathbb{NSK} \cap \mathbb{SK}^{(f)}$, and let $\mathbb{A}^{(f)}$ be the subgroup of $\mathbb{A}$ generated by symbols of the form $[a]$, where $a \in \frac{1}{f}\mathbb{Z}/\mathbb{Z}$. Let $\mathbb{U}^{(f)}$ be the quotient of $\mathbb{A}^{(f)}$ by the subgroup generated by all elements of $\mathbb{A}^{(f)}$ of the form $[a] - \sum_{pb=a}[b]$, where p is a prime number dividing f and $a \in \frac{p}{f}\mathbb{Z}/\mathbb{Z}$. Then it can be shown that there exist canonical isomorphisms

$$\begin{aligned} H^2(\pm, \mathbb{U}^{(f)}) &= H_0(H_0(\mathbb{SK}^{(f)}, \partial), \delta) \\ &= H_0(\mathbb{SK}^{(f)}, \partial + \delta) = H_0(\mathbb{SK}^{(f)}/\mathbb{NSK}^{(f)}, \partial + \delta), \\ H^1(\pm, \mathbb{U}^{(f)}) &= H_{-1}(H_0(\mathbb{SK}^{(f)}, \partial), \delta) \\ &= H_{-1}(\mathbb{SK}^{(f)}, \partial + \delta) = H_{-1}(\mathbb{SK}^{(f)}/\mathbb{NSK}^{(f)}, \partial + \delta). \end{aligned}$$

Since direct limits commute with homology, by taking direct limits over the set of all odd integers we have the following canonical isomorphisms:

$$\begin{aligned} H^2(\pm, \mathbb{U}^{(\text{odd})}) &= H_0(H_0(\mathbb{SK}^{(\text{odd})}, \partial), \delta) \\ &= H_0(\mathbb{SK}^{(\text{odd})}, \partial + \delta) = H_0(\mathbb{SK}^{(\text{odd})}/\mathbb{NSK}^{(\text{odd})}, \partial + \delta), \\ H^1(\pm, \mathbb{U}^{(\text{odd})}) &= H_{-1}(H_0(\mathbb{SK}^{(\text{odd})}, \partial), \delta) \\ &= H_{-1}(\mathbb{SK}^{(\text{odd})}, \partial + \delta) = H_{-1}(\mathbb{SK}^{(\text{odd})}/\mathbb{NSK}^{(\text{odd})}, \partial + \delta), \end{aligned}$$

where $\mathbb{SK}^{(\text{odd})}$ is the subgroup of $\mathbb{SK}$ generated by all symbols of the form $[a, g, j]$, where j is any integer, g is odd, positive and squarefree, and $a \in \mathbb{Q}/\mathbb{Z}$, with the denominator of $\langle a \rangle$ odd. The symbols $\mathbb{NSK}^{(\text{odd})}$ and $\mathbb{U}^{(\text{odd})}$ are defined similarly. In the sequel we will often consider the following situation:

Let $\mathbf{a} = \sum m_i [a_i] \in H^2(\pm, \mathbb{U})$, and let f be the lcm of the denominators of the $\langle a_i \rangle$. For simplicity we will assume that f is odd. Let C be a cycle in $\mathbb{SK}$ such that

$$C = \bigoplus_{i+j=0} C_{i,j}, \quad \text{with } C_{0,0} = \sum m_i [a_i, 1, 0] \text{ and } (\partial + \delta)C = 0.$$

With this hypothesis, the above discussion shows that we may assume that C is a cycle in the subcomplex $\mathbb{SK}^{(\text{odd})}$ of the double complex $\mathbb{SK}$.

4. Some homological results for the double complex $\mathbb{SK}$

In this section, we define a vertical shift operator S, acting on the double complex $\mathbb{SK}$. For all primes p, we also define a diagonal shift operator Δ_p, acting on the double complex $\mathbb{SK}$. We compute the anticommutator of S with the differential $(\partial + \delta)$ of the total complex. We also demonstrate that for each prime p, $(\partial + \delta)$ commutes with Δ_p. These results form the building blocks of our structure theorems for algebraic gamma monomials.

Definition 3. We define the *vertical shift operator* $S\colon \mathbb{SK}_{m,n} \to \mathbb{SK}_{m,n+1}$, by the rule

$$S : [a, g, n] \mapsto (-1)^m [a, g, n+1],$$

where $m =$ number of prime factors of g.

Note that S is Λ-linear and induces the map $S\colon Tot_{m+n}(\mathbb{SK}) \to Tot_{m+n+1}(\mathbb{SK})$.

Theorem 4. *Let S be the vertical shift operator defined above. Let ∂ and δ be the differentials for $\mathbb{SK}$ of bidegrees $(-1, 0)$ and $(0, -1)$ respectively (as defined earlier). Note that $\partial + \delta : Tot_{m+n}(\mathbb{SK}) \to Tot_{m+n-1}(\mathbb{SK})$ is the differential of degree -1 for the total complex of $\mathbb{SK}$. Then the anticommutators of S with δ, ∂ and $(\partial + \delta)$ are as follows:*

$$\delta S + S\delta = 2, \qquad \partial S + S\partial = 0, \qquad (\partial + \delta)S + S(\partial + \delta) = 2.$$

Proof. It is enough to prove these relations for the operators in question acting on any generator $[a, g, n] \in \mathbb{SK}_{m,n}$, where m is the number of prime factors of g. From the definitions of δ and S we have

$$\begin{aligned} \delta S [a, g, n] &= \delta(-1)^m [a, g, n+1] \\ &= (-1)^m(-1)^m([a, g, n] - (-1)^{n+1} [-a, g, n]) \\ &= [a, g, n] + (-1)^n [-a, g, n]. \end{aligned}$$

Similarly,

$$S\delta\,[a,g,n] = [a,g,n] - (-1)^n\,[-a,g,n]\,.$$

It follows that $\delta S + S\delta = 2$. Next, from the definitions of ∂ and S we have:

$$\partial S\,[a,g,n] = \partial(-1)^m\,[a,g,n+1] = (-1)^m \sum_{p|g} \epsilon(g,p)(1-X_p)\,[a,g/p,n+1]\,.$$

Similarly,

$$S\partial\,[a,g,n] = (-1)^{m-1} \sum_{p|g} \epsilon(g,p)(1-X_p)\,[a,g/p,n+1]\,.$$

It follows that $\partial S + S\partial = 0$. The last relation is obvious. □

Definition 4. Given any prime p, we define the *diagonal shift operator* $\Delta_p\colon \mathbb{SK}_{m,n} \to \mathbb{SK}_{m-1,n+2}$ by the following rule:

For any $[a,g,n] \in \mathbb{SK}_{m,n}$, where m is the number of prime factors of g, set

$$\Delta_p\colon\ [a,g,n] \mapsto 0, \quad \text{if} \quad p \nmid g.$$

If $p \mid g$, then $p = p_r$ (say), where the prime factorisation of g is

$$g = p_1 p_2 \cdots p_r \cdots p_m, \quad \text{with} \quad p_1 < p_2 < \cdots < p_r < \cdots < p_m.$$

In this case set

$$\Delta_p\colon\ [a,g,n] \mapsto (-1)^{n+m-r}\,[a,g/p,n+2]\,.$$

Note that Δ_p is Λ-linear and induces the map

$$\Delta_p\colon Tot_{m+n}(\mathbb{SK}) \to Tot_{m+n+1}(\mathbb{SK}).$$

Theorem 5. *For any prime p, let Δ_p be the diagonal shift operator defined above. Let ∂ and δ be the differentials for $\mathbb{SK}$ of bidegrees $(-1,0)$ and $(0,-1)$ respectively (as defined earlier). Note that $\partial + \delta\colon Tot_{m+n}(\mathbb{SK}) \to Tot_{m+n-1}(\mathbb{SK})$ is the differential of degree -1 for the total complex of $\mathbb{SK}$. Then Δ_p commutes with ∂, δ and $(\partial+\delta)$, i.e.,*

$$\partial\Delta_p = \Delta_p\partial, \qquad \delta\Delta_p = \Delta_p\delta, \qquad (\partial+\delta)\Delta_p = \Delta_p(\partial+\delta).$$

Proof. It is enough to prove these relations for the operators in question acting on any generator $[a,g,n] \in \mathbb{SK}_{m,n}$. The proof is obvious if $p \nmid g$. If $p \mid g$, then $p = p_r$ (say), where the prime factorisation of g is $g = p_1 p_2 \cdots p_r \cdots p_m$, with $p_1 < p_2 < \cdots < p_r < \cdots < p_m$. Then $\partial\Delta_p\,[a,g,n] = (-1)^{n+m-r}\partial\,[a,g/p,n+2]$. Therefore, from the definition of ∂,

(7)

$$\partial\Delta_p\,[a,g,n] = (-1)^{n+m-r}\Bigg(\sum_{i=1}^{r-1}(-1)^i(1-X_{p_i})\,[a,g/p_i p_r,n+2] + \sum_{j=r+1}^{m}(-1)^{j-1}(1-X_{p_j})\,[a,g/p_j p_r,n+2]\Bigg).$$

Again $\Delta_p \partial\,[a,g,n] = \Delta_{p_r} \sum_{i=1}^m (-1)^i (1 - X_{p_i})\,[a, g/p_i, n]$. Therefore,

$$\begin{aligned}\Delta_p \partial\,[a,g,n] = \Delta_{p_r} & \sum_{i=1}^{r-1} (-1)^i (1 - X_{p_i})\,[a, g/p_i, n] \\ & + \Delta_{p_r} (-1)^r (1 - X_{p_r})\,[a, g/p_r, n] \\ & + \Delta_{p_r} \sum_{j=r+1}^{m} (-1)^j (1 - X_{p_j})\,[a, g/p_j, n]\,. \end{aligned} \tag{8}$$

Hence from the definition of Δ_{p_r}, we have

$$\begin{aligned}\Delta_p \partial\,[a,g,n] = (-1)^{n+m-1-(r-1)} & \sum_{i=1}^{r-1} (-1)^i (1 - X_{p_i})\,[a, g/p_i p_r, n+2] \\ & + 0 \\ & + (-1)^{n+m-1-r} \sum_{j=r+1}^{m} (-1)^j (1 - X_{p_j})\,[a, g/p_j p_r, n+2]\,. \end{aligned} \tag{9}$$

Comparing (7) and (9), we have $\partial \Delta_p = \Delta_p \partial$. Again,

$$\delta \Delta_p\,[a,g,n] = (-1)^{n+m-r} \delta\,[a, g/p_r, n+2]\,.$$

Therefore,

$$\begin{aligned}&\delta \Delta_p\,[a,g,n] \\ &\quad = (-1)^{n+m-r} (-1)^{m-1} \Big([a, g/p_r, n+1] - (-1)^{n+2}\,[-a, g/p_r, n+1] \Big). \end{aligned} \tag{10}$$

Simplifying the above, we get

$$\begin{aligned}&\delta \Delta_p\,[a,g,n] \\ &\quad = (-1)^{n+2m-r-1} \Big([a, g/p_r, n+1] - (-1)^{n}\,[-a, g/p_r, n+1] \Big). \end{aligned} \tag{11}$$

Also $\Delta_p \delta\,[a,g,n] = \Delta_{p_r} (-1)^m ([a,g,n-1] - (-1)^n\,[-a,g,n-1])$. Therefore,

$$\begin{aligned}&\Delta_p \delta\,[a,g,n] \\ &\quad = (-1)^m (-1)^{n-1+m-r} \Big([a, g/p_r, n+1] - (-1)^{n}\,[-a, g/p_r, n+1] \Big). \end{aligned} \tag{12}$$

From (11) and (12) we have $\delta \Delta_p = \Delta_p \delta$. The last relation is obvious. □

5. The square of an algebraic gamma monomial

Given an element $\mathbf{a} \in \mathbb{A}$, we first define the gamma monomial and the sine monomial corresponding to $\mathbf{a}$, as follows:

Definition 5. Let $\mathbf{a} = \sum m_i\,[a_i] \in \mathbb{A}$ ($m_i \in \mathbb{Z}$, and $a_i \in \mathbb{Q}/\mathbb{Z}$). We define the corresponding *gamma monomial* to be

$$\Gamma(\mathbf{a}) = \prod_{i:\ a_i \neq 0} \left(\frac{\sqrt{2\pi}}{\Gamma(\langle a_i \rangle)} \right)^{m_i},$$

where $\langle a_i \rangle$ is the smallest positive rational number representing the class of a_i in $\mathbb{Q}/\mathbb{Z}$.

Definition 6. With notation as above, the *sine monomial* corresponding to $\mathbf{a}$ is defined to be

$$\sin \mathbf{a} = \prod_{i:\ a_i \not\equiv 0} (2 \sin \pi \langle a_i \rangle)^{m_i}.$$

If $\sum m_i\,[a_i, g_i, n] \in \mathbb{SK}_{m,n}$, where m is the number of prime divisors of each g_i, we will often write $\Gamma(\sum m_i\,[a_i, g_i, n])$ to mean the algebraic gamma monomial corresponding to $\mathbf{a} = \sum m_i\,[a_i]$. By $\sin(\sum m_i\,[a_i, g_i, n])$ we mean the sine monomial corresponding to $\mathbf{a} = \sum m_i\,[a_i]$. Using the above notation and the homological results of the preceding section, our goal here is to derive an expression for the square of an algebraic gamma monomial corresponding to some $\mathbf{a} \in H^2(\pm, \mathbb{U})$. We have the following:

Theorem 6. *Let $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$, and let f be the lcm of the denominators of the $\langle a_i \rangle$. Then*

$$\Gamma(\mathbf{a})^{2f} = \sqrt{r}(\sin \mathbf{a})^f,$$

where $r \in \mathbb{Q}$.

Proof. From the definition of $\Gamma(\mathbf{a})$ it follows that without loss of generality we may assume that, for all i, $a_i \not\equiv 0 \mod \mathbb{Z}$. We know that $\mathbf{a}$ forms the bidegree $(0,0)$ component of a cycle C in $\mathbb{SK}$ such that $C = \bigoplus_{i+j=0} C_{i,j}$, $\quad C_{0,0} = \sum m_i\,[a_i, 1, 0]$, and $(\partial + \delta)C = 0$. Let $C_{1,-1} = \sum n_i\,[b_i, p_i, -1]$. By Theorem 4,

$$(\partial + \delta)SC + S(\partial + \delta)C = 2C.$$

Since C is a cycle, $(\partial + \delta)SC = 2C$. Therefore $2C_{0,0} = \delta SC_{0,0} + \partial SC_{1,-1}$. Observe that $\Gamma(2C_{0,0}) = \Gamma(\mathbf{a})^2$.

Now from the definition of the differential ∂ and Gauss' functional equation for the classical gamma function, we have

$$\Gamma(\partial S\,[b_i, p_i, -1]) = p_i^{1/2 - \langle b_i \rangle}.$$

From the definition of the differential δ and Weierstrass' functional equation for the classical gamma function, we have

$$\Gamma(\delta S\,[a_i, 1, 0]) = 2 \sin \pi \langle a_i \rangle.$$

It follows from these observations that

$$\Gamma(\mathbf{a})^2 = \left(\prod p_i^{n_i (1/2 - \langle b_i \rangle)} \right) \sin \mathbf{a}.$$

Therefore $\Gamma(\mathbf{a})^{2f} = \sqrt{r}(\sin \mathbf{a})^f$, where $r = \prod p_i^{n_i (f - 2f \langle b_i \rangle)}$ is a rational number. ☐

6. The Koblitz-Ogus criterion for elements in $H^2(\pm, \mathbb{U})$

Let $\mathbb{A}$ be the free abelian group generated by the symbols $[a]$, where $a \in \mathbb{Q}/\mathbb{Z}$. Let $\mathbf{a} = \sum m_i\,[a_i] \in \mathbb{A}$.

Definition 7. We define the *internal sum* of $\mathbf{a}$ as follows:

$$\text{Internal Sum of } \mathbf{a} = \sum m_i \langle a_i \rangle.$$

Let f be the lcm of the denominators of the a_i, and let $t \in (\mathbb{Z}/f\mathbb{Z})^{\times}$. (Equivalently, let t be an integer relatively prime to the denominators of the a_i.) Note that $(\mathbb{Z}/f\mathbb{Z})^{\times}$ acts naturally by internal multiplication on the subgroup of $\mathbb{A}$ generated by the symbols $[a]$ such that $a \in \frac{1}{f}\mathbb{Z}/\mathbb{Z}$.

Definition 8. We define $\mathbf{a}^t$ to be the image of $\mathbf{a}$ under internal multiplication by t. Thus

$$\mathbf{a}^t = \sum m_i \, [ta_i] \, .$$

The following theorem is due to Koblitz and Ogus:

Theorem 7. *With notation as above,* $\Gamma(\mathbf{a})$ *is algebraic if*

$$\sum m_i \langle a_i \rangle = \sum m_i \langle t a_i \rangle \quad \textit{for all } t \in (\mathbb{Z}/f\mathbb{Z})^{\times}.$$

Remark. For a proof of this we refer to [4]. Note that the theorem says that any $\mathbf{a} \in \mathbb{A}$ gives rise to an algebraic gamma monomial if its internal sum is invariant under the action of $(\mathbb{Z}/f\mathbb{Z})^{\times}$. We will refer to this criterion for internal sums as the Koblitz-Ogus criterion. We also mention that the converse of this theorem is a major unsolved conjecture (Rohrlich's conjecture) in transcendental number theory. As mentioned earlier, Kubert (building on Sinnott's ideas) computed $H^*(\pm, \mathbb{U})$, where $\mathbb{U}$ is the universal ordinary distribution. Kubert showed that $H^2(\pm, \mathbb{U})$ can be identified with the torsion subgroup of $\mathbb{U}^-$, where $\mathbb{U}^-$ is the universal odd distribution. The following theorem follows from Kubert's result:

Theorem 8. *With notation as above,*

$$\mathbf{a} \in H^2(\pm, \mathbb{U}) \Leftrightarrow \sum m_i \langle a_i \rangle = \sum m_i \langle t a_i \rangle$$

for all $t \in (\mathbb{Z}/f\mathbb{Z})^{\times}$. *In other words,* $\mathbf{a} \in H^2(\pm, \mathbb{U})$ *if and only if* $\mathbf{a}$ *satisfies the Koblitz-Ogus internal sum criterion.*

Proof. For a proof of the sufficiency of the Koblitz-Ogus criterion in the above statement, we refer to [4]. The necessity of the Koblitz-Ogus criterion follows easily from the double complex. We sketch this below: Since $\mathbf{a} \in H^2(\pm, \mathbb{U})$, therefore $\mathbf{a}$ forms the bidegree $(0,0)$ component of a cycle C in $\mathbb{SK}$ such that $C = \bigoplus_{i+j=0} C_{i,j}$, $C_{0,0} = \sum m_i \, [a_i, 1, 0]$, and $(\partial + \delta)C = 0$. Write $C_{1,-1} = \sum n_i \, [b_i, p_i, -1]$. Let $t \in (\mathbb{Z}/f\mathbb{Z})^{\times}$, and let $C^t = \bigoplus_{i+j=0} C^t_{i,j}$ be the cycle obtained from C by internally multiplying by t. Thus

$$C^t_{0,0} = \sum m_i \, [ta_i, 1, 0] \quad \text{and} \quad C^t_{1,-1} = \sum n_i \, [tb_i, p_i, -1] \, .$$

From the definition of ∂, we have

$$\text{Internal sum of } \partial C_{1,-1} = \text{Internal sum of } \partial C^t_{1,-1} = \sum n_i \frac{p_i - 1}{2}.$$

From the definition of δ, we have

$$\text{Internal sum of } \delta C_{0,0} = 2 \sum m_i \langle a_i \rangle - \sum m_i,$$
$$\text{Internal sum of } \delta C^t_{0,0} = 2 \sum m_i \langle t a_i \rangle - \sum m_i.$$

Since $\delta C_{0,0} = -\partial C_{1,-1}$ and $\delta C^t_{0,0} = -\partial C^t_{1,-1}$, we have $\sum m_i \langle a_i \rangle = \sum m_i \langle t a_i \rangle$. □

Remark. The following observation about internal multiplication of cycles will be used often in the sequel. Let C and C^t be cycles in $\mathbb{SK}$, as in the proof of Theorem 8 above. From Theorems 1 and 2, it follows that internal multiplication acts trivially on cohomology. Therefore $C - C^t$ is a boundary in $\mathbb{SK}$. Hence there exists a chain B in $\mathbb{SK}$ such that $B = \bigoplus_{i+j=1} B_{i,j}$ and $(\partial + \delta)B = C - C^t$.

7. Kummer property of the extension $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a}))/\mathbb{Q}(\zeta_f)$

We first prove the following proposition, which is a consequence of the Koblitz-Ogus criterion. The proposition itself will prove to be extremely useful for our future results. Here we make use of it in proving that $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a}))$ is a Kummer extension of $\mathbb{Q}(\zeta_f)$.

Notation. In the following i will be used both as a subscript and as a complex number, and which it is will be clear from the context.

Proposition 3. *Let* $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$*, and let* f *be the lcm of the denominators of the* $\langle a_i \rangle$*. Assume that* $a_i \not\equiv 0 \mod \mathbb{Z}$*. Then*

$$2 \sum m_i \langle a_i \rangle = \sum m_i.$$

Therefore if $2 \nmid f$*, then* $\sum m_i \equiv 0 \mod 2$*.*

Proof. Internally multiplying by -1 and using the Koblitz-Ogus criterion, we have $\sum m_i \langle a_i \rangle = \sum m_i (1 - \langle a_i \rangle)$. Hence $\sum m_i = 2 \sum m_i \langle a_i \rangle$. Obviously if $2 \nmid f$, then $2 \mid \sum m_i$. □

As a consequence of the above we have the following:

Proposition 4. *Let* $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$ $(a_i \not\equiv 0 \mod \mathbb{Z})$*, let* f *be the lcm of the denominators of the* $\langle a_i \rangle$*, and assume that* f *is odd. Let* C *be a cycle in* $\mathbb{SK}$ *such that* $C = \bigoplus_{i+j=0} C_{i,j}$*, with* $C_{0,0} = \sum m_i\,[a_i, 1, 0]$ *and* $(\partial + \delta)C = 0$*. From Remark B following Theorem 3 we may assume that* C *is a cycle in the subcomplex* $\mathbb{SK}^{(odd)}$ *of the double complex* $\mathbb{SK}$*. Let*

$$C_{1,-1} = \sum n_i\,[b_i, p_i, -1] + \sum n'_j\,\left[b'_j, p'_j, -1\right],$$

where $p_i \equiv 3 \mod 4$*,* $p'_j \equiv 1 \mod 4$*. Then* $\sum n_i \equiv 0 \mod 2$*.*

Proof. We have $\delta C_{0,0} + \partial C_{1,-1} = 0$. Taking internal sums of both sides, we get

$$2 \sum m_i \langle a_i \rangle - \sum m_i + \sum n_i \frac{p_i - 1}{2} + \sum n'_j \frac{p'_j - 1}{2} = 0.$$

By Proposition 3 the first two terms cancel, and since $\frac{p_i - 1}{2} \equiv 1 \mod 2$ and $\frac{p'_j - 1}{2} \equiv 0 \mod 2$, we have $\sum n_i \equiv 0 \mod 2$. □

We are now ready to prove the following theorem of Deligne [3]:

Theorem 9. *Let* $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$*, let* f *be the lcm of the denominators of the* a_i*, and let* f *be odd. Then* $(\Gamma(\mathbf{a}))^{2f} \in \mathbb{Q}(\zeta_f)$*. Therefore,* $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a}))$ *is a Kummer extension of* $\mathbb{Q}(\zeta_f)$*.*

Proof. Recall that by definition

$$\Gamma(\mathbf{a}) = \prod_{i:\ a_i \not\equiv 0} \left(\frac{\sqrt{2\pi}}{\Gamma(\langle a_i \rangle)} \right)^{m_i}.$$

Therefore without loss of generality we may assume that, for all i, $a_i \not\equiv 0 \mod \mathbb{Z}$. We use the same notation as in Proposition 4 above. As in the proof of Theorem 6, we have

$$\Gamma(\mathbf{a})^2 = \left(\prod p_i^{n_i(1/2-\langle b_i\rangle)}\right)\left(\prod {p'_j}^{n'_j(1/2-\langle b'_j\rangle)}\right)\sin\mathbf{a}.$$

Since the denominators of b_i and b'_j divide f, and since $2 \nmid f$, it follows that

$$\Gamma(\mathbf{a})^{2f} = r\left(\prod p_i^{n_i(1/2)}\right)\left(\prod {p'_j}^{n'_j(1/2)}\right)\sin\mathbf{a},$$

where r is a rational number. Now $\sum m_i$ is even, by Proposition 3. Hence $\sin\mathbf{a} \in \mathbb{Q}(\zeta_f)$. Also, each ${p'_j}^{1/2} \in \mathbb{Q}(\zeta_f)$, since $p'_j \equiv 1 \mod 4$. Furthermore each $i{p_i}^{1/2} \in \mathbb{Q}(\zeta_f)$, since $p_i \equiv 3 \mod 4$. By Proposition 4, $\sum n_i$ is even, therefore $\prod p_i^{n_i(1/2)} \in \mathbb{Q}(\zeta_f)$. Hence $\Gamma(\mathbf{a})^{2f} \in \mathbb{Q}(\zeta_f)$. □

Remark. The Kummer property of the extension $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a}))/\mathbb{Q}(\zeta_f)$ was proved by Deligne in [3], using the theory of absolute Hodge cycles. Our proof, using the double complex, is constructive and relatively elementary.

8. Canonical lifting

In this section we prescribe an algorithm for computing $\mathbf{a} \in H^2(\pm, \mathbb{U})$, given any of the canonical basis classes of $H^2(\pm, \mathbb{U})$, indexed by a squarefree odd positive integer divisible by an even number of primes. The same algorithm also computes elements of $H^1(\pm, \mathbb{U})$, given any of the canonical basis classes of $H^1(\pm, \mathbb{U})$, indexed by a squarefree odd positive integer divisible by an odd number of primes. Since our method provides a uniquely determined cycle in the double complex $\mathbb{SK}$, lifting a given basis class, we call this the *canonical lifting* of the basis class. We often call the cycle thus obtained a *canonically lifted cycle*. The method is as follows:

Fix $[0, g, -n] \in \mathbb{SK}$, where g is a squarefree positive integer divisible by n odd primes, where n is even. We want to construct a cycle C in $\mathbb{SK}$ such that

$$C = \bigoplus_{i=0}^{n} C_{i,-i}, \quad C_{n,-n} = [0, g, -n], \quad (\partial + \delta)C = 0.$$

We compute C by a diagram chase through the double complex $\mathbb{SK}$. Note that if $[b, g/p, -n+1]$ appears in $C_{n-1,-n+1}$, then

$$\delta\,[b, g/p, -n+1] = -([b, g/p, -n] + [1-b, g/p, -n]).$$

Observe that there are two possibilites: (1) $\langle b\rangle < 1/2$ and $\langle 1-b\rangle > 1/2$, or (2) $\langle b\rangle > 1/2$ and $\langle 1-b\rangle < 1/2$. Since $\delta C_{n-1,-n+1} = -\partial C_{n,-n}$, we therefore prescribe that $C_{n-1,-n+1}$ be obtained by retaining all terms in $\partial C_{n,-n}$ with corresponding entries strictly less than $1/2$. In other words, suppose

$$\partial C_{n,-n} = \sum_i m_i\,[c_i, h, -n]\,,$$

where h is of the form g/p, for some prime factor p of g. Then for $C_{n-1,-n+1}$ we prescribe the lifting

$$C_{n-1,-n+1} = (-1)^n \sum_j m_j\,[c_j, h, -n+1]\,,$$

where the summation is over all j for which $\langle c_j \rangle < 1/2$. To compute C we now proceed inductively. Having constructed a lifting for $C_{k,-k}$, we write

$$\partial C_{k,-k} = \sum_i n_i \, [d_i, w, -k] \, ,$$

where w is a squarefree positive integer having $k-1$ prime factors (which are also prime factors of g). Then for $C_{k-1,-k+1}$ we prescribe the lifting

$$C_{k-1,-k+1} = (-1)^k \sum_j n_j \, [d_j, w, -k+1] \, ,$$

where the summation is over all j for which $\langle d_j \rangle < 1/2$. By construction, we thus obtain a unique cycle C with each entry strictly less than $1/2$. Note also that, by construction, the denominator of each entry divides g.

Remark. Note that the same construction applies to the lifting of basis classes of $H^1(\pm, \mathbb{U})$, i.e., when n (the number of prime divisors of g) is odd. Since these constructions will be extremely useful in the sequel, we summarise our results in the following proposition:

Proposition 5. *Let* $[0, g, -n] \in \mathbb{SK}$, *where* g *is a squarefree positive integer divisible by* n *odd primes. Also let* $\mathbb{SK}^{(<\frac{1}{2})}$ *be the subcomplex of* $\mathbb{SK}$ *generated by symbols of the form* $[a, h, i]$ *such that* $0 < \langle a \rangle < 1/2$. *Then there exists a unique cycle* C *in* $\mathbb{SK}$, *with* $C - C_{n,-n} \in \mathbb{SK}^{(<\frac{1}{2})}$, *such that*

$$C = \bigoplus_{i=0}^{n} C_{i,-i}, \quad C_{n,-n} = [0, g, -n] \, , \quad (\partial + \delta) C = 0.$$

We call C *the canonically lifted cycle, lifting the basis class* $[0, g, -n]$.

In the following section we will use canonical lifting to derive explicit formulae for the squares of algebraic gamma monomials. First we prove a general result for canonically lifted cycles.

Proposition 6. *Let* $\mathbf{a} = \sum m_i \, [a_i] \in H^2(\pm, \mathbb{U})$ $(a_i \not\equiv 0 \mod \mathbb{Z})$. *Let* f *be the lcm of the denominators of the* $\langle a_i \rangle$, *and let* f *be odd. Suppose* $\langle a_i \rangle < 1/2$ *for all* i *(this is true if* $\mathbf{a}$ *forms the bidegree* $(0,0)$ *component of a canonically lifted cycle). Then*

$$\sum m_i \langle a_i \rangle = 0 \quad \textit{and} \quad \sum m_i = 0.$$

Proof. Since $\langle a_i \rangle < 1/2$, therefore $\langle 2a_i \rangle = 2\langle a_i \rangle$. Hence, by the Koblitz-Ogus criterion, $\sum m_i \langle a_i \rangle = \sum 2 m_i \langle a_i \rangle$, so that $\sum m_i \langle a_i \rangle = 0$. Also, since $a_i \not\equiv 0$, from Proposition 3 we see that $\sum m_i = 2 \sum m_i \langle a_i \rangle = 0$. □

The following results about cycles in $\mathbb{SK}$ will often prove to be useful in later results.

Definition 9. Let $C = \bigoplus_{i+j=0} C_{i,j}$ be a cycle in $\mathbb{SK}$. Given a squarefree positive integer g divisible by k primes, we write $C_{k,-k}$ as

$$C_{k,-k} = \sum_i n_i \, [a_i, g, -k] + \sum_j m_j \, [b_j, g_j, -k] \, ,$$

where the g_j are squarefree positive integers divisible by k primes, such that $g \neq g_j$ for all j. With this notation, we define the *g-component* of $C_{k,-k}$ to be

$$C_{k,-k}^{\{g\}} = \sum_i n_i \, [a_i, g, -k] \, .$$

Remark. We use braces around g to differentiate this case from our earlier notation $C_{k,-k}^t$, which denotes internal multiplication by t. Note that we can express $C_{k,-k}$ as a sum of its g-components as $C_{k,-k} = \sum_g C_{k,-k}^{\{g\}}$. We have the following:

Proposition 7. *Let $C = \bigoplus_{i+j=0} C_{i,j}$ be a cycle in $\mathbb{SK}$. Given a squarefree positive integer g, let $C_{k,-k}^{\{g\}}$ be the g-component of $C_{k,-k}$, as described above. Then $C_{k,-k} = \sum_g C_{k,-k}^{\{g\}}$. For fixed g let $C_{k,-k}^{\{g\}} = \sum_i n_i \, [a_i, g, -k]$. Then, if k is even,*

$$\sum_i n_i \, [a_i] \in H^2(\pm, \mathbb{U}).$$

In particular, in this case if we write $C_{k,-k} = \sum m_j \, [b_j, g_j, -k]$, then

$$\sum m_j \, [b_j] \in H^2(\pm, \mathbb{U}).$$

Proof. The proof is an easy consequence of Theorem 5. By definition

$$\begin{aligned} C_{k,-k} &= \sum_i n_i \, [a_i, g, -k] + \sum_j m_j \, [b_j, g_j, -k] \\ &= C_{k,-k}^{\{g\}} + \sum_j m_j \, [b_j, g_j, -k] \, , \end{aligned}$$

where the g_j are squarefree positive integers, divisible by k primes, such that for all j, $g \neq g_j$. Let the prime factorisation of g be $g = p_1 p_2 \cdots p_k$, with $p_1 < p_2 < \cdots < p_k$. Consider the chain D in $\mathbb{SK}$ given by $D = \Delta_{p_1} \Delta_{p_2} \cdots \Delta p_k C$. Recall that Δ_p induces the map $\Delta_p \colon Tot_{m+n}(\mathbb{SK}) \to Tot_{m+n+1}(\mathbb{SK})$. Therefore we have $D = \bigoplus_{i+j=k} D_{i,j}$. By Theorem 5, $(\partial + \delta)\Delta_p = \Delta_p(\partial + \delta)$; therefore $(\partial + \delta)D = 0$. Thus D is a chain in $\mathbb{SK}$. Since $\Delta_p \, [b, h, n] = 0$, if $p \nmid h$, it follows that for all j, $\Delta_{p_1} \Delta_{p_2} \cdots \Delta p_k \, [b_j, g_j, -k] = 0$, since $g \neq g_j$. Therefore $D_{0,k} = \eta \sum_i n_i \, [a_i, 1, k]$, where $\eta = \pm 1$, and is determined by the product of the signs which appear from each application of Δ_{p_i} in $\Delta_{p_1} \Delta_{p_2} \cdots \Delta p_k \, [a_i, g, -k]$. From the definition of Δ_p we find that $\eta = (-1)^{0-k}(-1)^{1-(k-1)}(-1)^{2-(k-2)} \cdots (-1)^{k-1-(1)} = (-1)^{-k}$. In particular if k is even, then $\eta = 1$, and $D_{0,k} = \sum_i n_i \, [a_i, 1, k]$. From the theory of the double complex, since k is even, the cycle $D \in H_k(\mathbb{SK}, \partial + \delta) = H_0(\mathbb{SK}, \partial + \delta)$. Therefore from $D_{0,k}$ we read off $\sum_i n_i \, [a_i] \in H^2(\pm, \mathbb{U})$. The last statement of the proposition is obvious. $\square$

Along the same lines, we also have

Proposition 8. *Let $C = \bigoplus_{i+j=0} C_{i,j}$ be a cycle in $\mathbb{SK}$. Given a squarefree positive integer g, let $C_{k,-k}^{\{g\}}$ be the g-component of $C_{k,-k}$, as described above. Then $C_{k,-k} = \sum_g C_{k,-k}^{\{g\}}$. For fixed g let $C_{k,-k}^{\{g\}} = \sum_i n_i \, [a_i, g, -k]$. Then, if k is odd,*

$$\sum_i n_i \, [a_i] \in H^1(\pm, \mathbb{U}).$$

In particular, in this case if we write $C_{k,-k} = \sum m_j \, [b_j, g_j, -k]$, then

$$\sum m_j \, [b_j] \in H^1(\pm, \mathbb{U}).$$

Proof. The proof is exactly similar to the proof of Proposition 7 above. □

9. Using canonical lifting to determine $\Gamma(\mathbf{a})^2$

In this section we use canonical lifting to compute $\Gamma(\mathbf{a})^2$, where $\mathbf{a} \in H^2(\pm, \mathbb{U})$. We consider two separate cases as follows:

First case. We first consider the case when $\mathbf{a}$ is obtained by canonically lifting any of the basis classes of $H^2(\pm, \mathbb{U})$, indexed by a squarefree positive integer which is the product of exactly two odd primes, say p and q, where $p < q$. We use canonical lifting to construct the canonically lifted cycle C, lifting the basis class $k_{pq} = [0, pq, -2] \in \mathbb{SK}_{2,-2}/\mathbb{NSK}_{2,-2}$, of $H^2(\pm, \mathbb{U})$. We write $C = C_{0,0} \oplus C_{1,-1} \oplus C_{2,-2}$, where $C_{2,-2} = [0, pq, -2]$, and $p < q$ are odd primes. As explained in the last section, we compute C by a diagram chase through the double complex $\mathbb{SK}$. We have

$$\partial C_{2,-2} = \sum_{i=1}^{p-1} [i/p, q, -2] - \sum_{j=1}^{q-1} [j/q, p, -2] .$$

Since $\delta C_{1,-1} = -\partial C_{2,-2}$, we choose

$$C_{1,-1} = \sum_{i=1}^{\frac{p-1}{2}} [i/p, q, -1] - \sum_{j=1}^{\frac{q-1}{2}} [j/q, p, -1] .$$

Then

$$\begin{aligned} \delta C_{0,0} = -\partial C_{1,-1} = &\sum_{i=1}^{\frac{p-1}{2}} \left([i/p, 1, -1] - \sum_{k=0}^{q-1} \left[\frac{i/p + k}{q}, 1, -1 \right] \right) \\ &- \sum_{j=1}^{\frac{q-1}{2}} \left([j/q, 1, -1] - \sum_{l=0}^{p-1} \left[\frac{j/q + l}{p}, 1, -1 \right] \right) . \end{aligned} \tag{13}$$

We therefore choose

$$\begin{aligned} C_{0,0} = &\sum_{i=1}^{\frac{p-1}{2}} \left([i/p, 1, 0] - \sum_{k=0}^{\frac{q-1}{2}} \left[\frac{i/p + k}{q}, 1, 0 \right] \right) \\ &- \sum_{j=1}^{\frac{q-1}{2}} \left([j/q, 1, 0] - \sum_{l=0}^{\frac{p-1}{2}} \left[\frac{j/q + l}{p}, 1, 0 \right] \right) . \end{aligned} \tag{14}$$

From the expression for $C_{0,0}$, we read off

$$\mathbf{a}_{pq} = \sum_{i=1}^{\frac{p-1}{2}} \left([i/p] - \sum_{k=0}^{\frac{q-1}{2}} \left[\frac{i/p + k}{q} \right] \right) - \sum_{j=1}^{\frac{q-1}{2}} \left([j/q] - \sum_{l=0}^{\frac{p-1}{2}} \left[\frac{j/q + l}{p} \right] \right) .$$

Then, by the theory of the double complex, $\mathbf{a}_{pq}$ represents a non-trivial element in the torsion subgroup of $\mathbb{U}^-$, and corresponds by construction to k_{pq}. A straight forward computation gives

$$\Gamma(\partial S C_{1,-1}) = \frac{q^{(p-1)^2/8p}}{p^{(q-1)^2/8q}} .$$

Since $\Gamma(\mathbf{a})^2 = \Gamma(\partial SC_{1,-1}) \sin \mathbf{a}$ (as in Theorem 6), we have in this case

$$\Gamma(\mathbf{a}_{pq})^2 = \frac{q^{(p-1)^2/8p}}{p^{(q-1)^2/8q}} \sin \mathbf{a}_{pq}. \tag{15}$$

Example. As an example, we compute the canonically lifted cycle C_{15}, lifting the basis class $k_{15} = [0, 15, -2] \in \mathbb{SK}_{2,-2}/\mathbb{NSK}_{2,-2}$. From the formulae above, we have $C = C_{0,0} \oplus C_{1,-1} \oplus C_{2,-2}$, where

$$C_{0,0} = [1/3, 1, 0] - [4/15, 1, 0] - [1/5, 1, 0] + [2/15, 1, 0]\,, \tag{16}$$

$$C_{1,-1} = [1/3, 5, -1] - [1/5, 3, -1] - [2/5, 3, -1]\,, \tag{17}$$

$$C_{2,-2} = [0, 15, -2]\,. \tag{18}$$

From the expression for $C_{0,0}$, we read off

$$\mathbf{a}_{15} = [1/3] - [4/15] - [1/5] + [2/15]\,, \tag{19}$$

Then $\mathbf{a}_{15}$ represents a non-trivial element in the torsion subgroup of $\mathbb{U}^-$, and corresponds by construction to k_{15}. We also have

$$\Gamma(\mathbf{a}_{15})^2 = 3^{-2/5} 5^{1/6} \sin \mathbf{a}_{15}. \tag{20}$$

Note that each entry in $\mathbf{a}_{15}$ is strictly less than $1/2$, and that the internal sum of $\mathbf{a}$ is $\langle 1/3\rangle - \langle 4/15\rangle - \langle 1/5\rangle + \langle 2/15\rangle = 0$, as stated in Proposition 6.

Second case. Next, we consider the case when $\mathbf{a}$ is obtained by canonically lifting any of the basis classes of $H^2(\pm, \mathbb{U})$, indexed by an odd squarefree positive integer divisible by at least four odd primes. We want to prove the following:

Theorem 10. *Let $C = \bigoplus_{i+j=0} C_{i,j}$ be a canonically lifted cycle in the double complex $\mathbb{SK}$, lifting any basis class of $H^2(\pm, \mathbb{U})$ represented by $k_g = [0, g, -n] \in \mathbb{SK}_{n,-n}/\mathbb{NSK}_{n,-n}$, where g is a squarefree odd positive integer divisible by an even number of primes. Let $C_{0,0} = \sum m_i\,[a_i, 1, 0]$ and $\mathbf{a} = \sum m_i\,[a_i]$. Then $\mathbf{a}$ represents the basis class k_g. If g is divisible by at least four odd primes, then*

$$\Gamma(\partial SC_{1,-1}) = 1, \quad \textit{hence} \quad \Gamma(\mathbf{a})^2 = \sin \mathbf{a}.$$

Proof. We express $C_{2,-2}$ as a sum of pq-components, where p and q are odd prime divisors of g with $p < q$. Using Definition 9, we write $C_{2,-2} = \sum_{(p,q)\colon\, p<q} C_{2,-2}^{\{pq\}}$. Let $C_{2,-2}^{\{pq\}} = \sum_i n_i\,[b_i, pq, -2]$. By Proposition 7

$$\sum_i n_i\,[b_i] \in H^2(\pm, \mathbb{U}). \tag{21}$$

To avoid having to use diamond braces repeatedly for all our formulae below, without loss of generality, in this proof we assume that for all i the number b_i, $0 \le b_i < 1$, represents the class of $b_i \in \mathbb{Q}/\mathbb{Z}$. Thus for this proof we write $\langle b_i\rangle = b_i$.

Since C is a canonically lifted cycle, by Proposition 5 we have $0 < b_i < 1/2$ for all i. Therefore, by Proposition 6

$$\sum_i n_i b_i = 0 \quad \text{and} \quad \sum_i n_i = 0. \tag{22}$$

Using canonical lifting, we compute $C_{1,-1}$ by a diagram chase through the double complex $\mathbb{SK}$. We proceed by determining the contribution of each $C_{2,-2}^{\{pq\}}$ to $C_{1,-1}$.

We have

$$\partial C^{\{pq\}}_{2,-2} = \sum_i \left(-n_i\,[b_i, q, -2] + n_i \sum_{\nu=0}^{p-1} \left[\frac{b_i+\nu}{p}, q, -2\right]\right) + \sum_i \left(n_i\,[b_i, p, -2] - n_i \sum_{\mu=0}^{q-1} \left[\frac{b_i+\mu}{q}, p, -2\right]\right). \tag{23}$$

Recall that for canonical lifting we retain terms strictly less than 1/2, to find the contribution to $C_{1,-1}$. We let $C_{1,-1}(pq)$ denote this contribution to $C_{1,-1}$, arising from $C^{\{pq\}}_{2,-2}$. Thus

$$C_{1,-1}(pq) = \sum_i \left(-n_i\,[b_i, q, -1] + n_i \sum_{\nu=0}^{\frac{p-1}{2}} \left[\frac{b_i+\nu}{p}, q, -1\right]\right) + \sum_i \left(n_i\,[b_i, p, -1] - n_i \sum_{\mu=0}^{\frac{q-1}{2}} \left[\frac{b_i+\mu}{q}, p, -1\right]\right). \tag{24}$$

Then from the above we have

$$\Gamma(\partial SC_{1,-1}(pq)) = \prod_i \left(\frac{q^{n_i \sum_{\nu=0}^{\frac{p-1}{2}} (1/2-(b_i+\nu)/p)}}{q^{n_i(1/2-b_i)}} \right) = \prod_i \left(\frac{p^{n_i(1/2-b_i)}}{p^{n_i \sum_{\mu=0}^{\frac{q-1}{2}} (1/2-(b_i+\mu)/q)}} \right). \tag{25}$$

Summing over ν and μ in the above expression, we get

$$\begin{aligned}\Gamma(\partial SC_{1,-1}(pq)) &= \prod_i \left(\frac{q^{n_i(p-1)^2/8p}}{p^{n_i(q-1)^2/8q}} \right) \left(\frac{q^{n_i b_i(p-1)/2p}}{p^{n_i b_i(q-1)/2q}} \right) \\ &= \left(\frac{q^{(\sum_i n_i)(p-1)^2/8p}}{p^{(\sum_i n_i)(q-1)^2/8q}} \right) \left(\frac{q^{(\sum_i n_i b_i)(p-1)/2p}}{p^{(\sum_i n_i b_i)(q-1)/2q}} \right) \\ &= 1,\end{aligned}$$

since, all the exponents are 0, by equations (21) and (22). Clearly $C_{1,-1} = \sum_{(p,q)\,:\,p<q} C_{1,-1}(pq)$, where we sum over all the contributions to $C_{1,-1}$ arising from the canonical lifting of each pq-component $C^{\{pq\}}_{2,-2}$, of $C_{2,-2}$. Since $\Gamma(\partial SC_{1,-1}) = \sum_{(p,q)\,:\,p<q} \Gamma(\partial SC_{1,-1}(pq))$, therefore $\Gamma(\partial SC_{1,-1}) = 1$. Clearly then, $\Gamma(\mathbf{a})^2 = \sin \mathbf{a}$. □

Remark. Canonical lifting is also useful in other computations involving the double complex. For example, in later sections we will often encounter the following problem: Given a boundary C in $\mathbb{SK}$, construct a chain B such that $(\partial + \delta)B = C$. In this situation, canonical lifting provides a natural method for lifting C to a unique chain B. Details are as follows:

Assume that $C = C_{0,s} \oplus C_{1,s-1} \oplus \cdots \oplus C_{n,s-n}$, is a boundary in $\mathbb{SK}$. Also for simplicity, assume that no terms of the form $[0, g, s-i]$ appear in any of the $C_{i,s-i}$. We need to construct a chain $B = B_{0,s+1} \oplus B_{1,s+1} \oplus \cdots \oplus B_{n,s-n+1}$ such that $(\partial + \delta)B = C$. As in our discussion of canonical lifting in Section 8, we describe the construction of B inductively. Thus $B_{n,s-n+1}$ is obtained by retaining all terms in $C_{n,s-n}$ with corresponding entries strictly less than $1/2$. We note that $\delta B_{i-1,s-i+2} + \partial B_{i,s-i+1} = C_{i-1,s-i+1}$. Therefore, having constructed $B_{i,s-i+1}$, we obtain $B_{i-1,s-i+2}$ by retaining all terms in $C_{i-1,s-i+1} - \partial B_{i,s-i+1}$ with entries strictly less than $1/2$. We summarise our observations in the following proposition, which is similar to Proposition 5.

Proposition 9. *Let $C = \bigoplus_{i=0}^{n} C_{i,s-i}$ be a boundary in $\mathbb{SK}$. Assume that no terms of the form $[0, g, s-i]$, appear in any of the $C_{i,s-i}$. Then there exists a unique chain B in $\mathbb{SK}^{(<\frac{1}{2})}$ such that*

$$B = \bigoplus_{i+j=s+1} B_{i,j}, \quad \textit{with} \quad (\partial + \delta)B = C,$$

where $\mathbb{SK}^{(<\frac{1}{2})}$ is the subcomplex of $\mathbb{SK}$ generated by symbols of the form $[a, h, i]$ such that $0 < \langle a\rangle < 1/2$. We call B the canonically lifted chain lifting the boundary C.

10. The ratio $\sin \mathbf{a}/\sin \mathbf{a}^t$

Let $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$ $(a_i \notin \mathbb{Z})$. Let f be the lcm of the denominators of the $\langle a_i\rangle$. We assume that f is odd for simplicity. From Proposition 3, we know that $\sum m_i$ is even. Since each $\sin \pi\langle a_i\rangle = (\zeta_{2f}^{f\langle a_i\rangle} - \zeta_{2f}^{-f\langle a_i\rangle})/2i$, hence $\sin \mathbf{a} \in \mathbb{Q}(\zeta_{2f}) = \mathbb{Q}(\zeta_f)$, since f is odd. Now, let $\mathbf{a}$ be as above and let $t \in (\mathbb{Z}/f\mathbb{Z})^\times$. Recall that, by definition, $\mathbf{a}^t = \sum m_i\,[ta_i]$. Our goal is to compute the ratio of the sine monomials corresponding to $\mathbf{a}$ and $\mathbf{a}^t$. For this we need the following propositions.

Proposition 10. *Let $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$. Then $\mathbf{a}$ forms the bidegree $(0,0)$ component of a cycle C in $\mathbb{SK}$ such that $C = \bigoplus_{i+j=0} C_{i,j}$, $C_{0,0} = \sum m_i\,[a_i, 1, 0]$, and $(\partial+\delta)C = 0$. Let C^t be the cycle obtained from C by internally multiplying by t, so that $\mathbf{a}^t$ is the bidegree $(0,0)$ component of C^t. Then $C - C^t$ is a boundary in $\mathbb{SK}$. Let B be the chain in $\mathbb{SK}$ such that $B = \bigoplus_{i+j=1} B_{i,j}$, and $(\partial+\delta)B = C - C^t$. Let S be the vertical shift operator as defined earlier. Then*

$$\Gamma(\partial S \delta B_{1,0}) = \Gamma(\partial B_{1,0})^2.$$

Note that the remark following Theorem 8 shows that $C - C^t$ is a boundary in $\mathbb{SK}$.

Proof. It is enough to show this for any $[b, p, 0] \in B_{1,0}$. We have

$$\begin{aligned}\partial S \delta\,[b,p,0] &= \partial S(-\,[b,p,-1] + [-b,p,-1]) \\ &= -(1-X_p)\,[b,1,0] + (1-X_p)\,[-b,1,0]\end{aligned}$$

Therefore

$$\Gamma(\partial S\delta\,[b,p,0]) = p^{-(1/2-\langle b\rangle)} p^{1/2-(1-\langle b\rangle)} = p^{2\langle b\rangle -1}.$$

Again,

$$\Gamma(\partial\,[b,p,0])^2 = \Gamma(-(1-X_p)\,[b,p,0])^2 = (p^{-(1/2-\langle b\rangle)})^2 = p^{2\langle b\rangle-1}.$$

This proves the proposition. □

The following is a direct consequence of the above:

Proposition 11. *Let* $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$. *Then* $\mathbf{a}$ *forms the bidegree* $(0,0)$ *component of a cycle* C *in* $\mathbb{SK}$ *such that* $C = \bigoplus_{i+j=0} C_{i,j}$, $C_{0,0} = \sum m_i\,[a_i, 1, 0]$, *and* $(\partial+\delta)C = 0$. *Let* C^t *be the cycle obtained from* C *by internally multiplying by* t, *so that* $\mathbf{a}^t$ *is the bidegree* $(0,0)$ *component of* C^t. *Let* B *be the chain in* $\mathbb{SK}$ *such that* $B = \bigoplus_{i+j=1} B_{i,j}$, *and* $(\partial+\delta)B = C - C^t$. *Let* S *be the vertical shift operator as defined earlier. Then*

$$\Gamma(\partial(S(C - C^t))_{1,0}) = \Gamma(\partial B_{1,0})^2.$$

Proof. Observe that $\partial S(C - C^t) = \partial S(\partial+\delta)B = \partial S \delta B$, since $\partial S = -S\partial$ and $\partial^2 = 0$. Therefore $\Gamma(\partial(S(C - C^t))_{1,0}) = \Gamma(\partial S \delta B_{1,0}) = \Gamma(\partial B_{1,0})^2$, by Proposition 10. This proves the proposition. ☐

We are now ready for the following theorem:

Theorem 11. *Let* $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$. *Let* f *be the lcm of the denominators of the* a_i, *and let* $t \in (\mathbb{Z}/f\mathbb{Z})^\times$. *We assume that* f *is odd for simplicity. Then there exists* $\mathbf{b} = \sum n_i\,[b_i] \in \mathbb{A}$ *such that*

$$\frac{\sin \mathbf{a}}{\sin \mathbf{a}^t} = (\sin \mathbf{b})^2$$

Furthermore, $\sin \mathbf{b} \in \mathbb{Q}(\zeta_f)$ *if* $\sum n_i$ *is even, and* $\sin \mathbf{b} \in \mathbb{Q}(\zeta_{4f})$ *if* $\sum n_i$ *is odd.*

Proof. We know that $\mathbf{a}$ forms the bidegree $(0,0)$ component of a cycle C in $\mathbb{SK}$ such that $C = \bigoplus_{i+j=0} C_{i,j}$, $C_{0,0} = \sum m_i\,[a_i, 1, 0]$, and $(\partial+\delta)C = 0$. Let C^t be the cycle obtained from C by internally multiplying by t, so that $\mathbf{a}^t$ is the bidegree $(0,0)$ component of C^t. Let B be the chain in $\mathbb{SK}$ such that $B = \bigoplus_{i+j=1} B_{i,j}$ and $(\partial+\delta)B = C - C^t$. Let S be the vertical shift operator as defined earlier. Since $(\partial+\delta)C = 0$ and $(\partial+\delta)S + S(\partial+\delta) = 2$, we have $(\partial+\delta)(C - C^t) = 2(C - C^t)$. Therefore $2C_{0,0} - 2C^t_{0,0} = \delta(S(C - C^t))_{0,1} + \partial(S(C - C^t))_{1,0}$. This gives an identity for $2\mathbf{a} - 2\mathbf{a}^t$ in $\mathbb{A}$. Hence

$$\frac{\Gamma(\mathbf{a})^2}{\Gamma(\mathbf{a}^t)^2} = \Gamma(\partial(S(C - C^t))_{1,0}) \frac{\sin \mathbf{a}}{\sin \mathbf{a}^t}. \tag{26}$$

Now let $B_{0,1} = \sum n_i\,[b_i, 1, 1]$, and write $\mathbf{b} = \sum n_i\,[b_i]$. We have $C_{0,0} - C^t_{0,0} = \delta B_{0,1} + \partial B_{1,0}$. This gives an identity for $\mathbf{a} - \mathbf{a}^t$ in $\mathbb{A}$. Hence

$$\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{a}^t)} = \Gamma(\partial B_{1,0}) \sin \mathbf{b} \tag{27}$$

From (26), (27), and Proposition 11 above, we have $\frac{\sin \mathbf{a}}{\sin \mathbf{a}^t} = (\sin \mathbf{b})^2$. For the last part of the theorem, observe that no term of the form $[0, g, -i]$ can appear in the boundary $C - C^t$. By Proposition 9, there exists a unique canonically lifted chain B in $\mathbb{SK}$ such that $B = \bigoplus_{i+j=1} B_{i,j}$, with $(\partial+\delta)B = C$. As before, we write $B_{0,1} = \sum n_i\,[b_i, 1, 1]$ and $\mathbf{b} = \sum n_i\,[b_i]$. Then, by construction, for all i we have $b_i \not\equiv 0 \mod \mathbb{Z}$, and the denominators of the b_i divide f. Recall that by definition, since $b_i \not\equiv 0 \mod \mathbb{Z}$, we have $\sin \mathbf{b} = \prod_i (2 \sin \pi\langle b_i\rangle)^{n_i}$. The last part of the theorem is now clear from the fact that $\sin \pi\langle b_i\rangle = (\zeta_{2f}^{f\langle b_i\rangle} - \zeta_{2f}^{-f\langle b_i\rangle})/2i$. ☐

Example. To illustrate Theorem 11, we turn to our example of the canonically lifted cycle C_{15}, lifting the basis class $[0, 15, -2]$, of $H^2(\pm, \mathbb{U})$. Let $t = 7$. From equation (16), we have

$$\begin{aligned} C^7_{0,0} &= [1/3,1,0] - [13/15,1,0] - [2/5,1,0] + [14/15,1,0]\,, \\ C^7_{1,-1} &= [1/3,5,-1] - [2/5,3,-1] - [4/5,3,-1]\,, \\ C^7_{2,-2} &= [0,15,-2]\,. \end{aligned}$$

Note that $(C - C^7)_{2,-2} = 0$. We want to find a chain B such that $B = B_{0,1} \oplus B_{1,0}$, with $(\partial + \delta)B = C - C^7$. Since $(C - C^7)_{1,-1} = -[1/5,3,-1] + [4/5,3,-1]$, we use canonical lifting to conclude that $B_{1,0} = [1/5,3,0]$. Now

$$\delta B_{0,1} = (C - C^7)_{0,0} - \partial B_{1,0}.$$

A straightforward calculation, using canonical lifting gives $B_{0,1} = [2/15,1,1] - [1/15,1,1] - [4/15,1,1]$. Thus

$$\begin{aligned} \mathbf{a} &= [1/3] - [4/15] - [1/5] + [2/15]\,, \\ \mathbf{a}^7 &= [1/3] - [13/15] - [2/5] + [14/15]\,, \\ \mathbf{b} &= [2/15] - [1/15] - [4/15]\,, \end{aligned}$$

and we have the identity $\sin \mathbf{a} / \sin \mathbf{a}^7 = (\sin \mathbf{b})^2$. Explicitly,

$$\begin{aligned} &\left(\frac{\sin(\pi/3)\sin(2\pi/15)}{\sin(4\pi/15)\sin(\pi/5)}\right)\left(\frac{\sin(13\pi/15)\sin(2\pi/5)}{\sin(\pi/3)\sin(14\pi/15)}\right) \\ &\quad = \left(\frac{\sin(2\pi/15)}{2\sin(\pi/15)\sin(4\pi/15)}\right)^2. \end{aligned} \tag{28}$$

11. Relation between $\sin \mathbf{a}^t$ and $\sigma_t \sin \mathbf{a}$

Let $\sigma_t \in \mathrm{Gal}(\mathbb{Q}(\zeta_{2f}))$. Here, as always, ζ_{2f} stands for the primitive $2f$-th root of unity, and $\sigma_t : \zeta_{2f} \mapsto \zeta_{2f}^t$. Note that t is an odd integer, relatively prime to f. We want to find the relationship between $\sin \mathbf{a}^t$ and $\sigma_t \sin \mathbf{a}$. First note that $\sin \mathbf{a}^t$ is a positive real number, since each $\langle t a_i \rangle$ lies between 0 and 1. However, for $\mathbf{a} \in \mathbb{A}$ with $\sin \mathbf{a} \in \mathbb{Q}(\zeta_f)$, it is quite possible that $\sigma_t \sin \mathbf{a} = -\sin \mathbf{a}^t$. For example, if $\mathbf{a} = [1/3] + [1/5]$, then $\sigma_7 \sin \mathbf{a} = 2\sin(7\pi/3)2(\sin 7\pi/5) = -\sin \mathbf{a}^7$. However, if $\mathbf{a} \in \mathbb{A}$ is an element of $H^2(\pm, \mathbb{U})$ then we have the following:

Theorem 12. *Let $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$. Let f be the lcm of the $\langle a_i \rangle$ and let f be odd. Let t be an odd integer relatively prime to f, and let $\sigma_t : \zeta_{2f} \mapsto \zeta_{2f}^t$. Then*

$$\sigma_t \sin \mathbf{a} = \sin \mathbf{a}^t$$

Proof. Recall that by definition $\sin \mathbf{a} = \prod_{i:\, a_i \not\equiv 0} (2 \sin \pi \langle a_i \rangle)^{m_i}$. Therefore without loss of generality we may assume that, for all i, $a_i \not\equiv 0 \mod \mathbb{Z}$. Write $t\langle a_i \rangle = \lfloor t\langle a_i \rangle \rfloor + \langle t\langle a_i \rangle \rangle = \lfloor t\langle a_i \rangle \rfloor + \langle t a_i \rangle$. Then $t \sum m_i \langle a_i \rangle = \sum m_i \lfloor t\langle a_i \rangle \rfloor + \sum m_i \langle t a_i \rangle = \sum m_i \lfloor t\langle a_i \rangle \rfloor + \sum m_i \langle a_i \rangle$. In the last step we have used the Koblitz-Ogus criterion for $\mathbf{a} \in H^2(\pm, \mathbb{U})$. Thus $(t-1)\sum m_i \langle a_i \rangle = \sum m_i \lfloor t\langle a_i \rangle \rfloor$. Since t is odd, $\sum m_i \lfloor t\langle a_i \rangle \rfloor \equiv 0 \mod 2$. The proof follows from the fact that

$$\begin{aligned} \sigma_t \sin \mathbf{a} &= \prod (2 \sin \pi t \langle a_i \rangle)^{m_i} = \prod (2 \sin \pi (\lfloor t\langle a_i \rangle \rfloor + \langle t a_i \rangle))^{m_i} \\ &= \prod (2(-1)^{\lfloor t\langle a_i \rangle \rfloor} \sin \pi \langle t a_i \rangle)^{m_i} = (-1)^{\sum m_i \lfloor t\langle a_i \rangle \rfloor} \prod (2 \sin \pi \langle t a_i \rangle)^{m_i} \\ &= \prod (2 \sin \pi \langle t a_i \rangle)^{m_i} \text{ (by above)} = \sin \mathbf{a}^t. \end{aligned}$$

□

Example. To illustrate Theorem 12, once again we look at our example with C_{15}. We have

$$\sin \mathbf{a} = \frac{\sin(\pi/3)\sin(2\pi/15)}{\sin(4\pi/15)\sin(\pi/5)} \quad \text{and} \quad \sin \mathbf{a}^7 = \frac{\sin(\pi/3)\sin(14\pi/15)}{\sin(13\pi/15)\sin(2\pi/5)}.$$

Thus

$$\sigma_7 \sin \mathbf{a} = \frac{\sin(7\pi/3)\sin(14\pi/15)}{\sin(28\pi/15)\sin(7\pi/5)} = \frac{\sin(\pi/3)\sin(14\pi/15)}{(-\sin(13\pi/15))(-\sin(2\pi/5))} = \sin \mathbf{a}^7.$$

12. The ratio $(\Gamma(\mathbf{a})/\Gamma(\mathbf{a}^t))^f$

We have seen that $\Gamma(\mathbf{a})^{2f} \in \mathbb{Q}(\zeta_f)$. The same is true for $\Gamma(\mathbf{a}^t)^{2f}$. However in general neither $\Gamma(\mathbf{a})^f$ nor $\Gamma(\mathbf{a}^t)^f$ is in $\mathbb{Q}(\zeta_f)$. Here we want to prove the important fact that the ratio $(\Gamma(\mathbf{a})/\Gamma(\mathbf{a}^t))^f$ always belongs to $\mathbb{Q}(\zeta_f)$. In fact, we have the following:

Theorem 13. *Let $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$. Let f be the lcm of the $\langle a_i \rangle$ and let f be odd. Let t be an odd integer relatively prime to f, and let $\sigma_t : \zeta_{2f} \mapsto \zeta_{2f}^t$. Then*

$$\left(\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{a}^t)}\right)^f \in \mathbb{Q}(\zeta_f).$$

Proof. Recall that by definition $\Gamma(\mathbf{a}) = \prod_{i:\, a_i \not\equiv 0} \left(\frac{\sqrt{2\pi}}{\Gamma(\langle a_i \rangle)}\right)^{m_i}$. Therefore without loss of generality we may assume that, for all i, $a_i \not\equiv 0 \mod \mathbb{Z}$. We know that $\mathbf{a}$ forms the bidegree $(0,0)$ component of a cycle C in $\mathbb{SK}$ such that $C = \bigoplus_{i+j=0} C_{i,j}$, $C_{0,0} = \sum m_i\,[a_i, 1, 0]$ and $(\partial + \delta)C = 0$. Let C^t be the cycle obtained from C by internally multiplying by t, so that $\mathbf{a}^t$ is the bidegree $(0,0)$ component of C^t. Note that no term of the form $[0, g, -i]$ appears in the boundary $C - C^t$. By Proposition 9, there exists a unique canonically lifted chain B in $\mathbb{SK}$ such that $B = \bigoplus_{i+j=1} B_{i,j}$ with $(\partial + \delta)B = C$. As before, we write $B_{0,1} = \sum n_i\,[b_i, 1, 1]$ and $\mathbf{b} = \sum n_i\,[b_i]$. Then by construction, for all i, $b_i \not\equiv 0 \mod \mathbb{Z}$, and the denominators of the b_i divide f. We also write $B_{1,0} = \sum \epsilon_i\,[c_i, p_i, 0] + \sum \epsilon'_j\,[c'_j, p'_j, 0]$, where $p_i \equiv 3 \mod 4$, $p'_j \equiv 1 \mod 4$ and $\epsilon_i, \epsilon'_j = \pm 1$. We have $\mathbf{a} - \mathbf{a}^t = \delta B_{0,1} + \partial B_{1,0}$. From the Koblitz-Ogus criterion we know that the internal sum of the left hand side of the above equation is 0. On the right hand side, the internal sum of $\delta B_{0,1}$ is $\sum n_i$. Also, the internal sum of $\partial B_{1,0}$ is

$$\sum \epsilon_i \frac{p_i - 1}{2} + \sum \epsilon'_j \frac{p'_j - 1}{2}.$$

Equating internal sums of both sides, we obtain

$$\sum n_i + \sum \epsilon_i \frac{p_i - 1}{2} + \sum \epsilon'_j \frac{p'_j - 1}{2} = 0.$$

Since $(p_i - 1)/2 \equiv 1 \mod 2$ and $(p'_j - 1)/2 \equiv 0 \mod 2$, we have

$$\sum n_i + \sum \epsilon_i \equiv 0 \mod 2. \tag{29}$$

Now

$$\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{a}^t)} = \left(\prod p_i^{\epsilon_i(1/2 - \langle c_i \rangle)}\right)\left(\prod {p'_j}^{\epsilon'_j(1/2 - \langle c'_j \rangle)}\right) \sin \mathbf{b}, \tag{30}$$

so that

$$\left(\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{a}^t)}\right)^f = q\left(\prod p_i^{\epsilon_i(1/2)}\right)\left(\prod {p'_j}^{\epsilon'_j(1/2)}\right)(\sin \mathbf{b})^f,$$

where q is a rational number. Note that each ${p'_j}^{1/2} \in \mathbb{Q}(\zeta_f)$, since $p'_j \equiv 1 \mod 4$. Also $ip_i^{1/2} \in \mathbb{Q}(\zeta_f)$, since $p_i \equiv 3 \mod 4$. Therefore, if $\sum \epsilon_i$ is even, then $\prod p_i^{\epsilon_i(1/2)} \in \mathbb{Q}(\zeta_f)$. If $\sum \epsilon_i$ is odd, then $i \prod p_i^{\epsilon_i(1/2)} \in \mathbb{Q}(\zeta_f)$.

Again, if $\sum n_i$ is even, then $\sin \mathbf{b} \in \mathbb{Q}(\zeta_f)$. If $\sum n_i$ is odd, then $i \sin \mathbf{b} \in \mathbb{Q}(\zeta_f)$. By the congruence (29) above, $\sum n_i$ is even if and only if $\sum \epsilon_i$ is even. Combining these observations, we conclude that $(\Gamma(\mathbf{a})/\Gamma(\mathbf{a}^t))^f \in \mathbb{Q}(\zeta_f)$. □

Example. Once again, as an example to illustrate Theorem 13, we look at C_{15}. For $t = 7$ we have $B_{1,0} = [1/5, 3, 0]$. Also $B_{0,1} = [2/15, 1, 1] - [1/15, 1, 1] - [4/15, 1, 1]$. We conclude that

$$\left(\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{a^7})}\right)^{15} = 3^{-9/2}\left(\frac{\sin(2\pi/15)}{2\sin(\pi/15)\sin(4\pi/15)}\right)^{15}.$$

Note that $i3^{1/2} \in \mathbb{Q}(\zeta_{15})$. Also

$$i\left(\frac{\sin(2\pi/15)}{2\sin(\pi/15)\sin(4\pi/15)}\right) \in \mathbb{Q}(\zeta_{15}).$$

Therefore $\left(\Gamma(\mathbf{a})/\Gamma(\mathbf{a}^7)\right)^{15} \in \mathbb{Q}(\zeta_{15})$.

13. Criterion for an element of $\mathbb{A}$ to be in $H^2(\pm, \mathbb{U})$

For an element of $\mathbb{A}$ to be in $H^1(\pm, \mathbb{U})$, the Koblitz-Ogus criterion requires internal sums to be invariant under the action of $(\mathbb{Z}/f\mathbb{Z})^\times$. Here we prove a similar, necessary criterion for an element of $\mathbb{A}$ to be in $H^1(\pm, \mathbb{U})$. This criterion will prove to be useful for our results on double coverings in the following section. Our criterion requires that the corresponding sine monomial be invariant under the action of $(\mathbb{Z}/f\mathbb{Z})^\times$. First we prove a general result about elements in $H^1(\pm, \mathbb{U})$.

Proposition 12. *Let* $\mathbf{b} = \sum m_i\,[b_i] \in H^1(\pm, \mathbb{U})$*. Then, for all* i*,* $b_i \not\equiv 0 \mod \mathbb{Z}$*.*

Proof. This holds since there exists a cycle C in $\mathbb{SK}$ such that $C = \bigoplus_{i+j=1} C_{i,j}$, $C_{0,1} = \sum m_i\,[b_i, 1, 1]$, and $(\partial + \delta)C = 0$. If $[0, 1, 1]$ appears in $C_{0,1}$, then $\delta\,[0, 1, 1] = 2\,[0, 1, 0]$. But for any $[a, p, 0]$ in $C_{1,0}$, $\partial\,[a, p, 0]$ does not contain terms of the form $[0, 1, 0]$. Thus C would fail to be a cycle. □

We want to prove the following necessary criterion for an element of $\mathbb{A}$ to be in $H^1(\pm, \mathbb{U})$:

Theorem 14. *Let* $\mathbf{b} = \sum m_i\,[b_i] \in H^1(\pm, \mathbb{U})$*. Let* d *be the lcm of the denominators of the* b_i*. We assume* d *is odd for simplicity. Then, for all* $t \in (\mathbb{Z}/d\mathbb{Z})^\times$*, we have*

$$\sin \mathbf{b} = \sin \mathbf{b}^t = w.$$

That is, $\prod(2\sin\pi\langle b_i\rangle)^{m_i} = \prod(2\sin\pi\langle tb_i\rangle)^{m_i} = w$*, where* w *is independent of* t*. Furthermore we have the following cases:*

First case: When $\mathbf{b}$ *represents any of the canonical basis classes of* $H^1(\pm, \mathbb{U})$*, indexed by a single odd prime, then* $w = \sqrt{q}$*, where* $q \in \mathbb{Q}$*.*

Second case: Here let $\mathbf{b}$ *represent any of the canonical basis classes of* $H^1(\pm, \mathbb{U})$*, indexed by an odd squarefree positive integer divisible by at least three primes. Let* C

be a cycle in $\mathbb{SK}$ *such that* $C = \bigoplus_{i+j=1} C_{i,j}$, $C_{0,1} = \sum m_i\,[b_i, 1, 1]$, *and* $(\partial+\delta)C = 0$. *Assume that no term of the form* $[0, p, 0]$ *appears in* $C_{1,0}$. *Then* $w = 1$.

Proof. Note that by Proposition 12, $b_i \not\equiv 0 \mod \mathbb{Z}$.

First Case: In this case, there is a cycle C in $\mathbb{SK}$ such that $C = \bigoplus_{i+j=1} C_{i,j}$, $C_{0,1} = \sum m_i\,[b_i, 1, 1]$, and $(\partial + \delta)C = 0$. Let $C_{1,0} = \sum n_i\,[0, p_i, 0]$. Note that $C_{i,1-i} = 0$ for $i \geq 2$. Let C^t be the cycle obtained from C by internal multiplication by t. Then $C^t_{0,1} = \sum m_i\,[tb_i, 1, 0]$ and $C^t_{1,0} = C_{1,0}$. Therefore $\Gamma(-\partial C^t_{1,0}) = \Gamma(-\partial C_{1,0}) = \prod (\sqrt{p_i})^{n_i}$. Again, $\delta C = -\partial C$ (since C is a cycle); hence $\Gamma(\delta C^t_{0,1}) = \Gamma(\delta C_{0,1}) = \prod (\sqrt{p_i})^{n_i}$.

But

$$\Gamma(\delta C_{0,1}) = \Gamma\left(\sum m_i([b_i, 1, 0] + [-b_i, 1, 0])\right) = \prod (2 \sin \pi\langle b_i\rangle)^{m_i}.$$

Similarly

$$\Gamma(\delta C^t_{0,1}) = \Gamma\left(\sum m_i([tb_i, 1, 0] + [-tb_i, 1, 0])\right) = \prod (2 \sin \pi\langle tb_i\rangle)^{m_i}.$$

Writing $\prod {p_i}^{n_i} = q \in \mathbb{Q}$, we get

$$\prod (2 \sin \pi\langle b_i\rangle)^{m_i} = \prod (2 \sin \pi\langle tb_i\rangle)^{m_i} = w,$$

where $w = \sqrt{q}$, is a constant.

Second Case: Now assume that $\mathbf{b}$ lifts canonical basis classes indexed by odd squarefree positive integers divisible by at least three primes. By hypothesis, $C_{1,0}$ consists of terms of the form $[a, p, 0]$, where $a \not\equiv 0 \mod \mathbb{Z}$. We express $C_{1,0}$ as a sum of its p-components as $C_{1,0} = \sum_p C^{\{p\}}_{1,0}$. Let $C^{\{p\}}_{1,0} = \sum_i n_i\,[a_i, p, 0]$. Then by Proposition 7 $\sum_i n_i\,[a_i] \in H^2(\pm, \mathbb{U})$. Since $a_i \not\equiv 0 \mod \mathbb{Z}$, by Proposition 3 we have $\sum n_i = 2 \sum n_i\langle a_i\rangle$. Therefore $\Gamma(-\partial C^{\{p\}}_{1,0}) = p^{\sum n_i(1/2-\langle a_i\rangle)} = p^0 = 1$. Hence $\Gamma(-\partial C_{1,0}) = \prod_p \Gamma(-\partial C^{\{p\}}_{1,0}) = 1$. Again, $C^t_{1,0} = \sum_p (C^{\{p\}}_{1,0})^t$, where $(C^{\{p\}}_{1,0})^t = \sum_i n_i\,[ta_i, p, 0]$. But $\sum_i n_i\,[ta_i] \in H^2(\pm, \mathbb{U})$. (This is obvious from the Koblitz-Ogus criterion, since $\sum n_i\langle ta_i\rangle = \sum n_i\langle a_i\rangle$.) Also note that $ta_i \not\equiv 0 \mod \mathbb{Z}$. Hence (as above) we have $\Gamma(-\partial C^t_{1,0}) = 1$. Using $\Gamma\delta C_{0,1} = -\Gamma\partial C_{1,0}$ and $\Gamma\delta C^t_{0,1} = -\Gamma\partial {C^t}_{1,0}$, we get $\prod (2 \sin \pi\langle b_i\rangle)^{m_i} = \prod (2 \sin \pi\langle tb_i\rangle)^{m_i} = 1$. □

14. Some consequences of Theorem 14

In this section we derive some important properties of elements of $\mathbb{A}$ which belong to $H^1(\pm, \mathbb{U})$. These are direct consequences of the criterion described in Theorem 14. The results described here, will be extremely useful for our discussion of double coverings in the following section. First we have the following:

Proposition 13. *Let* $\mathbf{b} = \sum m_i[b_i] \in H^1(\pm, \mathbb{U})$. *Assume that* $\mathbf{b}$ *represents any of the canonical basis classes of* $H^1(\pm, \mathbb{U})$, *indexed by odd squarefree positive integers divisible by at least three odd primes. Let* C *be a cycle in* $\mathbb{SK}$ *such that* $C = \bigoplus_{i+j=1} C_{i,j}$, $C_{0,1} = \sum m_i\,[b_i, 1, 1]$, *and* $(\partial + \delta)C = 0$. *Assume that no term of the form* $[0, p, 0]$ *appears in* $C_{1,0}$. *Let* d *be the lcm of the denominators of the* b_i, *and let* d *be odd. Then*

$$\sum m_i \equiv 0 \mod 2.$$

Proof. Note that by Proposition 12, $b_i \not\equiv 0 \mod \mathbb{Z}$. By hypothesis, $C_{1,0}$ consists of terms of the form $[a, p, 0]$, where $a \not\equiv 0 \mod \mathbb{Z}$. We express $C_{1,0}$ as a sum of its p-components as $C_{1,0} = \sum_p C_{1,0}^{\{p\}}$. Let $C_{1,0}^{\{p\}} = \sum_i n_i\,[a_i, p, 0]$. By Proposition 7, $\sum_i n_i\,[a_i] \in H^2(\pm, \mathbb{U})$. Since $a_i \not\equiv 0 \mod \mathbb{Z}$, by Proposition 3 we have

$$\sum n_i \equiv 0 \mod 2. \tag{31}$$

Now $\delta C_{0,1} = -\partial C_{1,0}$. Therefore

$$\text{internal sum of } \delta C_{0,1} = \sum m_i = \text{ internal sum of } -\partial C_{1,0}. \tag{32}$$

But internal sum of $-\partial C_{1,0}^{\{p\}} = \sum_i n_i \frac{p-1}{2} = \frac{p-1}{2}(\sum_i n_i) \equiv 0 \mod 2$, by (31). Therefore $-\partial C_{1,0} = \sum_p -\partial C_{1,0}^{\{p\}} \equiv 0 \mod 2$. Hence by (32), $\sum m_i \equiv 0 \mod 2$. □

The following proposition, is a direct consequence of the criterion for elements in $H^1(\pm, \mathbb{U})$, given by Theorem 14.

Proposition 14. *Let* $\mathbf{b} = \sum m_i[b_i] \in H^1(\pm, \mathbb{U})$. *Assume that* $\mathbf{b}$ *represents any of the canonical basis classes of* $H^1(\pm, \mathbb{U})$, *indexed by odd squarefree positive integers divisible by at least three odd primes. Let* C *be a cycle in* $\mathbb{SK}$ *such that* $C = \bigoplus_{i+j=1} C_{i,j}$, $C_{0,1} = \sum m_i\,[b_i, 1, 1]$, *and* $(\partial + \delta)C = 0$. *Assume that no term of the form* $[0, p, 0]$ *appears in* $C_{1,0}$. *Let* d *be the lcm of the denominators of the* b_i, *and let* d *be odd. Let* t *be an odd integer relatively prime to* d, *and let* $\sigma_t : \zeta_{2d} \to \zeta_{2d}^t$. *Then*

$$\sin \mathbf{b} \in \mathbb{Q}(\zeta_d), \quad \textit{and} \quad \sigma_t \sin \mathbf{b} = \sin \mathbf{b} = \sin \mathbf{b}^t = 1.$$

Proof. Note that, by Proposition 12, $b_i \not\equiv 0 \mod \mathbb{Z}$. By Proposition 13, $\sum m_i$ is even. Hence $\sin \mathbf{b} \in \mathbb{Q}(\zeta_d)$. But from Theorem 14, since $\mathbf{b}$ represents a canonical basis class indexed by at least three odd primes, we have $\sin \mathbf{b} = \sin \mathbf{b}^t = 1$. The proof of the proposition is now obvious. □

Proposition 15. *Let* $\mathbf{b} = \sum m_i[b_i] \in H^1(\pm, \mathbb{U})$. *Assume that* $\mathbf{b}$ *represents any of the canonical basis classes of* $H^1(\pm, \mathbb{U})$, *indexed by odd squarefree positive integers divisible by at least three odd primes. Let* C *be a cycle in* $\mathbb{SK}$ *such that* $C = \bigoplus_{i+j=1} C_{i,j}$, $C_{0,1} = \sum m_i\,[b_i, 1, 1]$, *and* $(\partial + \delta)C = 0$. *Assume that no term of the form* $[0, p, 0]$ *appears in* $C_{1,0}$. *Let* d *be the lcm of the denominators of the* b_i, *and let* d *be odd. Let* t *be an odd integer relatively prime to* d. *Then*

$$d \sum m_i(\langle tb_i \rangle - \langle b_i \rangle) \equiv 0 \mod 2.$$

Proof. Note that by Proposition 12, $b_i \not\equiv 0 \mod \mathbb{Z}$. We have

$$\begin{aligned} \sigma_t \sin \mathbf{b} &= 1 \quad \text{by Proposition 14} \\ &= \prod (2 \sin \pi t \langle b_i \rangle)^{m_i} = \prod (2 \sin \pi (\lfloor t \langle b_i \rangle \rfloor + \langle tb_i \rangle))^{m_i} \\ &= (-1)^{\sum m_i \lfloor t \langle b_i \rangle \rfloor} \sin \mathbf{b}^t = (-1)^{\sum m_i \lfloor t \langle b_i \rangle \rfloor}, \quad \text{by Proposition 14.} \end{aligned}$$

Therefore $\sum m_i \lfloor t \langle b_i \rangle \rfloor = \sum m_i (t \langle b_i \rangle - \langle tb_i \rangle) \equiv 0 \mod 2$. So $d \sum m_i(\langle tb_i \rangle - \langle b_i \rangle) \equiv d(t-1) \sum m_i \langle b_i \rangle \bmod 2$. The proposition follows, since t is odd. □

Propositions 13 and 14 provides useful information about the ratio

$$\Gamma(\mathbf{a})^{2f} / (\sin \mathbf{a})^f,$$

when $\mathbf{a}$ represents any of the canonical basis classes of $H^2(\pm, \mathbb{U})$, indexed by odd squarefree positive integers divisible by at least four primes. Recall that, without any assumption on the basis classes, Theorem 6 states that this ratio is the square root of a rational number. We are now ready to prove the following generalisation of Theorem 6 and Theorem 10:

Theorem 15. *Let* $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$ $(a_i \notin \mathbb{Z})$, *and let* f *be the lcm of the denominators of the* $\langle a_i \rangle$ *and let* f *be odd. Then the following are true:*

$$\text{(A)} \qquad \Gamma(\mathbf{a})^{2f} = \sqrt{r}(\sin \mathbf{a})^f, \quad \textit{where } r \in \mathbb{Q}.$$

(This is true even without the assumption that f *is odd.)*

(B) *Let* $\mathbf{a}$ *represent any of the basis classes of* $H^2(\pm, \mathbb{U})$, *indexed by odd squarefree positive integers divisible by at least four primes. Let* C *be a cycle in* $\mathbb{SK}$ *such that* $C = \bigoplus_{i+j=o} C_{i,j}$, $C_{0,0} = \sum m_i\,[a_i, 1, 0]$, *and* $(\partial + \delta)C = 0$. *Assume that no term of the form* $[0, pq, 0]$ *appears in* $C_{2,-2}$. *Then*

$$\Gamma(\mathbf{a})^{2f} = s(\sin \mathbf{a})^f, \quad \textit{where } s \in \mathbb{Q}.$$

(C) *Let* t *be an odd integer relatively prime to* f. *If* $\mathbf{a}$ *forms the bidegree* $(0,0)$ *component of a cycle obtained by canonically lifting any of the basis classes of* $H^2(\pm, \mathbb{U})$, *indexed by odd squarefree positive integers divisible by at least four primes, then*

$$\Gamma(\mathbf{a})^{2f} = (\sin \mathbf{a})^f \quad \textit{and} \quad \Gamma(\mathbf{a}^t)^{2f} = w^2(\sin \mathbf{a}^t)^f, \quad \textit{where } w \in \mathbb{Q}.$$

Proof. (**A**) This is a restatement of Theorem 6.

(**B**) We express $C_{1,-1}$ as a sum of its p-components, $C_{1,-1} = \sum_p C_{1,-1}^{\{p\}}$. Let $C_{1,-1}^{\{p\}} = \sum_i n_i\,[b_i, p, -1]$. Then, by Proposition 8, $\sum_i n_i\,[b_i] \in H^1(\pm, \mathbb{U})$. By hypothesis, $\sum_i n_i\,[b_i]$ represents canonical basis classes of $H^1(\pm, \mathbb{U})$, indexed by odd squarefree positive integers divisible by at least three odd primes. Also the hypothesis, of Proposition 13 is satisfied, since $C_{2,-2}$ does not contain terms of the form $[0, pq, -2]$. Hence, from Proposition 13, $\sum_i n_i \equiv 0 \mod 2$. Now

$$\Gamma(\mathbf{a})^{2f} = (\Gamma(\partial S C_{1,-1}))^f (\sin \mathbf{a})^f = \prod_p (\Gamma(\partial S C_{1,-1}^{\{p\}}))^f (\sin \mathbf{a})^f,$$

$$(\Gamma(\partial S C_{1,-1}^{\{p\}}))^f = p^{f \sum_i n_i (1/2 - \langle b_i \rangle)} = p^{e(p)}, \quad \text{say.}$$

Since $\sum_i n_i$ is even, therefore $e(p)$ is an integer. The result follows from the fact that $\Gamma(\mathbf{a})^{2f} = \prod p^{e(p)} (\sin \mathbf{a})^f$.

(**C**) We use the same notation as in the proof of (**B**), and assume that C is a canonically lifted cycle. We then have

$$\Gamma(\mathbf{a})^{2f} = (\Gamma(\partial S C_{1,-1}))^f (\sin \mathbf{a})^f = \prod_p (\Gamma(\partial S C_{1,-1}^{\{p\}}))^f (\sin \mathbf{a})^f,$$

$$\Gamma(\mathbf{a}^t)^{2f} = (\Gamma(\partial S C_{1,-1}^t))^f (\sin \mathbf{a}^t)^f = \prod_p (\Gamma(\partial S (C_{1,-1}^{\{p\}})^t))^f (\sin \mathbf{a}^t)^f.$$

Furthermore,

$$(\Gamma(\partial S C_{1,-1}^{\{p\}}))^f = p^{f \sum_i n_i (1/2 - \langle b_i \rangle)} (\sin \mathbf{a})^f,$$

$$(\Gamma(\partial S (C_{1,-1}^{\{p\}})^t))^f = p^{f \sum_i n_i (1/2 - \langle t b_i \rangle)} (\sin \mathbf{a}^t)^f.$$

Since C is assumed to be a canonically lifted cycle, by Theorem 10 we have $(\Gamma(\partial SC_{1,-1}))^f = 1$. Now

$$\frac{(\Gamma(\partial SC_{1,-1}^{\{p\}}))^f}{(\Gamma(\partial S(C_{1,-1}^{\{p\}})^t))^f} = \frac{p^{f\sum_i n_i(1/2-\langle b_i\rangle)}}{p^{f\sum_i n_i(1/2-\langle tb_i\rangle)}} = p^{f\sum_i n_i(\langle tb_i\rangle - \langle b_i\rangle)} = p^{r(p)}, \quad \text{say.}$$

On the other hand, from Proposition 8, $\mathbf{b} = \sum_i n_i\,[b_i] \in H^1(\pm, \mathbb{U})$. Also, since C is a canonically lifted cycle, $C_{2,-2}$ does not contain elements of the form $[0, pq, -2]$. Hence the hypothesis of Proposition 15 holds. Therefore

$$r(p) = f\sum_i n_i(\langle tb_i\rangle - \langle b_i\rangle) \equiv 0 \mod 2.$$

Now,

$$\frac{(\Gamma(\partial SC_{1,-1}))^f}{(\Gamma(\partial S(C_{1,-1})^t))^f} = \prod_p p^{r(p)}.$$

Therefore

$$\frac{1}{(\Gamma(\partial S(C_{1,-1})^t))^f} = \prod_p p^{r(p)}.$$

Since $r(p)$ is even, $(\Gamma(\partial S(C_{1,-1})^t))^f = w^2$, where $w \in \mathbb{Q}$. This proves (**C**). □

15. Double coverings

Definition 10. Given a Galois extension K/F, we define a *double covering* of K/F to be an extension $\tilde{K}/K$ of degree ≤ 2 such that $\tilde{K}/F$ is Galois.

Let $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$, let f be the lcm of the denominators of the $\langle a_i\rangle$, and let f be odd. Then, by Theorems 11 and 12, for all $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ we have $\frac{\sqrt{\sin\mathbf{a}}}{\sigma\sqrt{\sin\mathbf{a}}} = \pm\sin\mathbf{b}$, where $\sin\mathbf{b} \in \mathbb{Q}(\zeta_{4f})$. Hence, $\mathbb{Q}(\zeta_{4f}, \sqrt{\sin\mathbf{a}})$ is a Galois extension of $\mathbb{Q}$, and so, by Definition 10, each $\mathbf{a} \in H^2(\pm, \mathbb{U})$ gives rise to a double covering of $\mathbb{Q}(\zeta_{4f})/\mathbb{Q}$ by $\mathbb{Q}(\zeta_{4f}, \sqrt{\sin\mathbf{a}})$. We want to prove the following theorem, due to Deligne:

Theorem 16. *Let $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$, let f be the lcm of the denominators of the $\langle a_i\rangle$, and let f be odd. Then, for all $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$,*

$$\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})} \in \mathbb{Q}(\zeta_f).$$

Thus $\mathbf{a}$ *gives rise to a double covering of* $\mathbb{Q}(\zeta_f)/\mathbb{Q}$ *by* $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a})^f)$.

Remark. Theorem 16 was proved by Deligne in [3], using the theory of absolute Hodge cycles. The double complex provides a more constructive and relatively elementary proof.

Notation. Let $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$, let f be the lcm of the denominators of the $\langle a_i\rangle$, and let f be odd. Let C be a cycle in $\mathbb{SK}$ such that $C = \bigoplus_{i+j=0} C_{i,j}$, $C_{0,0} = \sum m_i\,[a_i, 1, 0]$ and $(\partial + \delta)C = 0$. Furthermore let $C_{1,-1} = \sum \epsilon_i\,[b_i, p_i, -1]$. To simplify notation we write

$$v = \Gamma(\partial SC_{1,-1}) = \prod p_i^{\epsilon_i(1/2-\langle b_i\rangle)},$$
$$v_t = \Gamma(\partial SC^t{}_{1,-1}) = \prod p_i^{\epsilon_i(1/2-\langle tb_i\rangle)}.$$

With this notation we have $\Gamma(\mathbf{a})^2 = v \sin \mathbf{a}$ and $\Gamma(\mathbf{a}^t)^2 = v_t \sin \mathbf{a}^t$. Note that v^{2f}, ${v_t}^{2f}$ and $(\frac{v}{v_t})^f$ are rational numbers.

To prove Theorem 16, we first make the following reduction:

Theorem 17. *Let* $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$, *let* f *be the lcm of the denominators of the* $\langle a_i \rangle$, *and let* f *be odd. Let* $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, *and let* σ_t *be the restriction of* σ *to* $\mathbb{Q}(\zeta_f)$, *where, as usual,* $\sigma_t \colon \zeta_f \mapsto \zeta_f^t$. *Then*

$$\left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right) \in \mathbb{Q}(\zeta_f) \Leftrightarrow \frac{(v/v_t)^{f/2}}{(v/\sigma v)^{f/2}} \in \mathbb{Q}(\zeta_f).$$

Remark. Note that Theorem 17 provides a criterion for the verification of Theorem 16.

Proof. As in earlier proofs, we may assume that for all i, $a_i \not\equiv 0 \mod \mathbb{Z}$. From Theorem 13 (see (30)) we know that

$$\left(\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{a}^t)}\right)^f = \left(\frac{v}{v_t}\right)^{f/2} (\sin \mathbf{b})^f \in \mathbb{Q}(\zeta_f),$$

where $\mathbf{b} = \sum n_i b_i$, $b_i \not\equiv 0 \mod \mathbb{Z}$, and the denominators of the b_i divide f. Therefore Theorem 13 can be restated as

$$\begin{aligned} \left(\frac{v}{v_t}\right)^{f/2} &\in \mathbb{Q}(\zeta_f) \Leftrightarrow \sin \mathbf{b} \in \mathbb{Q}(\zeta_f), \\ i\left(\frac{v}{v_t}\right)^{f/2} &\in \mathbb{Q}(\zeta_f) \Leftrightarrow i \sin \mathbf{b} \in \mathbb{Q}(\zeta_f). \end{aligned} \tag{33}$$

On the other hand, we also have

$$\begin{aligned} \left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right)^f &= \sqrt{\frac{v^f}{\sigma v^f}} \left(\frac{\sin \mathbf{a}}{\sigma \sin \mathbf{a}}\right)^{f/2} \\ &= \sqrt{\frac{v^f}{\sigma v^f}} \left(\frac{\sin \mathbf{a}}{\sin \mathbf{a}^t}\right)^{f/2} \quad \text{by Theorem 12} \\ &= \pm\sqrt{\frac{v^f}{\sigma v^f}} (\sin \mathbf{b})^f \quad \text{by Theorem 11.} \end{aligned} \tag{34}$$

Furthermore, $v^f/\sigma v^f = \pm 1$, since v^f is the square root of a rational number. Therefore proving that $(\Gamma(\mathbf{a})/\sigma\Gamma(\mathbf{a}))^f \in \mathbb{Q}(\zeta_f)$, is equivalent to proving that

$$\begin{aligned} \frac{v^f}{\sigma v^f} &= 1 \Leftrightarrow \sin \mathbf{b} \in \mathbb{Q}(\zeta_f), \\ \frac{v^f}{\sigma v^f} &= -1 \Leftrightarrow i \sin \mathbf{b} \in \mathbb{Q}(\zeta_f). \end{aligned} \tag{35}$$

Clearly, (33) and (35) imply $\left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right)^f \in \mathbb{Q}(\zeta_f)$ if and only if the following is true:

$$\begin{aligned} \frac{v^f}{\sigma v^f} &= 1 \Leftrightarrow \left(\frac{v}{v_t}\right)^{f/2} \in \mathbb{Q}(\zeta_f), \\ \frac{v^f}{\sigma v^f} &= -1 \Leftrightarrow i\left(\frac{v}{v_t}\right)^{f/2} \in \mathbb{Q}(\zeta_f). \end{aligned}$$

Therefore to complete the proof of Theorem 17, we need to show that

$$\left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right)^f \in \mathbb{Q}(\zeta_f) \Leftrightarrow \left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right) \in \mathbb{Q}(\zeta_f).$$

We consider two possible cases:

First case: First let $v^f/\sigma v^f = 1$. Then $(\Gamma(\mathbf{a})/\sigma\Gamma(\mathbf{a}))^f = \pm(\sin\mathbf{b})^f$. Hence in this case

$$\left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right)^f \in \mathbb{Q}(\zeta_f) \Leftrightarrow \sin\mathbf{b} \in \mathbb{Q}(\zeta_f) \Leftrightarrow \left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right) \in \mathbb{Q}(\zeta_f),$$

since up to sign $(\Gamma(\mathbf{a})/\sigma\Gamma(\mathbf{a}))$ is the product of a primitive f-th root of unity and $\sin\mathbf{b}$.

Second Case: Now let $v^f/\sigma v^f = -1$. Then $(\Gamma(\mathbf{a})/\sigma\Gamma(\mathbf{a}))^f = \pm i(\sin\mathbf{b})^f$. Hence in this case

$$\left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right)^f \in \mathbb{Q}(\zeta_f) \Leftrightarrow i\sin\mathbf{b} \in \mathbb{Q}(\zeta_f) \Leftrightarrow \left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right) \in \mathbb{Q}(\zeta_f),$$

since, up to sign, $(\Gamma(\mathbf{a})/\sigma\Gamma(\mathbf{a}))$ is the product of a primitive f-th root of unity and $i\sin\mathbf{b}$. This proves the theorem. □

We need one further reduction for the proof of Theorem 16:

Proposition 16. *Let $k_g \in \mathbb{SK}_{i,-i}/\mathbb{NSK}_{i,-i}$ be a canonical basis class of $H^2(\pm,\mathbb{U})$, indexed by a squarefree odd positive integer g divisible by i primes, where i is even. Let C and C' be two different cycles lifting k_g—that is, C and C' both represent the same canonical basis class of $H^2(\pm,\mathbb{U})$. Let $\mathbf{a}$ and $\mathbf{a}'$ be obtained from the bidegree $(0,0)$ components of C and C' respectively, and let $\mathbf{a} = \sum m_i a_i$ and $\mathbf{a}' = \sum m_i a_i'$. Let f be the lcm of the denominators of the a_i and a_i', and assume that f is odd. Then*

$$\left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right) \in \mathbb{Q}(\zeta_f) \Leftrightarrow \left(\frac{\sigma\Gamma(\mathbf{a}')}{\Gamma(\mathbf{a}')}\right) \in \mathbb{Q}(\zeta_f).$$

Proof. There exists a canonically lifted chain B in $\mathbb{SK}$ such that $(\partial+\delta)B = C - C'$. As before, it follows that $(\Gamma(\mathbf{a})/\Gamma(\mathbf{a}'))^f = \sqrt{r}(\sin\mathbf{b})^f$, where r is a rational number, and $\mathbf{b}$ is identified with $B_{0,1}$. If $b = \sum n_i\,[b_i]$, then $\sin\mathbf{b} = \prod(2\sin\pi\langle b_i\rangle)^{n_i}$. Since B is a canonically lifted chain, therefore $b_i \not\equiv 0 \mod \mathbb{Z}$, and the lcm of the denominators of the b_i divide f. We have

$$\sin\pi\langle b_i\rangle = \frac{{\zeta_{2f}}^{f\langle b_i\rangle} - {\zeta_{2f}}^{-f\langle b_i\rangle}}{2i}.$$

If $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ extends $\sigma_t \in Gal(\mathbb{Q}(\zeta_{2f})/\mathbb{Q})$, then

$$\frac{\sin\pi\langle b_i\rangle}{\sigma\sin\pi\langle b_i\rangle} = \frac{\sigma i}{i}\frac{{\zeta_{2f}}^{f\langle b_i\rangle} - {\zeta_{2f}}^{-f\langle b_i\rangle}}{{\zeta_{2f}}^{tf\langle b_i\rangle} - {\zeta_{2f}}^{-tf\langle b_i\rangle}} = \pm\frac{\sin\pi\langle b_i\rangle}{\sin\pi\langle tb_i\rangle} \in \mathbb{Q}(\zeta_f).$$

Therefore $\sin\mathbf{b}/\sigma\sin\mathbf{b} \in \mathbb{Q}(\zeta_f)$. Also $\sqrt{r}/\sigma\sqrt{r} = \pm1$. Thus

$$\frac{\sqrt{r}(\sin\mathbf{b})^f}{\sigma(\sqrt{r}(\sin\mathbf{b})^f)} \in \mathbb{Q}(\zeta_f).$$

Therefore

$$\left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right)\left(\frac{\sigma\Gamma(\mathbf{a}')}{\Gamma(\mathbf{a}')}\right) \in \mathbb{Q}(\zeta_f),$$

so that

$$\left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right) \in \mathbb{Q}(\zeta_f) \Leftrightarrow \left(\frac{\sigma\Gamma(\mathbf{a}')}{\Gamma(\mathbf{a}')}\right) \in \mathbb{Q}(\zeta_f).$$

□

Proof of Theorem 16, First Case. To prove Theorem 16 we first consider the case when $\mathbf{a}$ represents any of the canonical basis classes of $H^2(\pm, \mathbb{U})$, indexed by a squarefree positive integer which is the product of exactly two odd primes, say p and q, where $p < q$. Our proof in this case involves Gauss' lemma from quadratic reciprocity. We know that $\mathbf{a}$ forms the bidegree $(0,0)$ component of a cycle C in $\mathbb{SK}$. We write $C = C_{0,0} \oplus C_{1,-1} \oplus C_{2,-2}$, where $C_{2,-2} = [0, pq, -2]$, and $p < q$ are odd primes. By Proposition 16 there is no loss of generality in assuming that C is a canonically lifted cycle. From our discussion of canonical lifting, we know that

$$C_{1,-1} = \sum_{i=1}^{\frac{p-1}{2}} [i/p, q, -1] - \sum_{j=1}^{\frac{q-1}{2}} [j/q, p, -1],$$

$$v = \Gamma(\partial S C_{1,-1}) = \prod_{i=1}^{\frac{p-1}{2}} q^{1/2 - i/p} \prod_{j=1}^{\frac{q-1}{2}} p^{-(1/2 - j/q)} = \frac{q^{(p-1)^2/8p}}{p^{(q-1)^2/8q}}.$$

Now if $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ extends $\sigma_t \in Gal(\mathbb{Q}(\zeta_{2pq})/\mathbb{Q})$, then from elementary number theory we know that $\frac{v^f}{\sigma v^f} = (t/q)^{\frac{p-1}{2}} (t/p)^{\frac{q-1}{2}}$, where $f = pq$ and (t/p) and (t/q) are the Legendre symbols of t with respect to p and q respectively. Since $\frac{p-1}{2}$ is even (respectively odd) if p is congruent to 1 (respectively 3) mod 4, we conclude from the above discussion that

$$\tag{36} \begin{aligned} \frac{v^f}{\sigma v^f} = 1 \quad & \text{if} \quad p \equiv q \equiv 1 \mod 4, \text{ or} \\ & \text{if} \quad p \equiv q \equiv 3 \mod 4 \quad \text{with} \quad (t/p)(t/q) = 1, \text{ or} \\ & \text{if} \quad p \equiv 1, q \equiv 3 \mod 4, \quad \text{with} \quad (t/p) = 1, \text{ or} \\ & \text{if} \quad p \equiv 3, q \equiv 1 \mod 4, \quad \text{with} \quad (t/q) = 1, \end{aligned}$$

and

$$\tag{37} \begin{aligned} \frac{v^f}{\sigma v^f} = -1 \quad & \text{if} \quad p \equiv q \equiv 3 \mod 4 \quad \text{with} \quad (t/p)(t/q) = -1, \text{ or} \\ & \text{if} \quad p \equiv 1, q \equiv 3 \mod 4, \quad \text{with} \quad (t/p) = -1, \text{ or} \\ & \text{if} \quad p \equiv 3, q \equiv 1 \mod 4, \quad \text{with} \quad (t/q) = -1. \end{aligned}$$

Now, to prove Theorem 16 we need to verify the equivalent criterion provided by Theorem 17. For this we need to compute $(v/v_t)^{f/2}$. We have

$$\tag{38} \begin{aligned} (C - C^t)_{1,-1} = & \sum_{i=1}^{\frac{p-1}{2}} ([i/p, q, -1] - [ti/p, q, -1]) \\ & - \sum_{j=1}^{\frac{q-1}{2}} ([j/q, p, -1] - [tj/q, p, -1]). \end{aligned}$$

Hence,

$$\left(\frac{v}{v_t}\right)^{f/2} = \frac{q^{f/2\sum_{1=i}^{\frac{p-1}{2}} i/p-\langle ti/p\rangle}}{p^{f/2\sum_{j=1}\frac{q-1}{2} j/q-\langle tj/q\rangle}}. \tag{39}$$

Finally, we need the following proposition to complete our proof of Theorem 16. Notice that this proposition is a direct consequence of Gauss' lemma from quadratic reciprocity. □

Proposition 17. *Let p be an odd prime and let t be an integer relatively prime to p. Then*

$$\begin{aligned} p\sum_{i=1}^{\frac{p-1}{2}}(i/p-\langle ti/p\rangle) &\equiv 0 \mod 2, \quad \textit{if} \quad (t/p)=1,\\ &\equiv 1 \mod 2, \quad \textit{if} \quad (t/p)=-1. \end{aligned}$$

Proof. We write $it/p = \lfloor it/p\rfloor + \langle it/p\rangle$, where $i = 1,\dots,\frac{p-1}{2}$. We also let $Q = \sum_{i=1}^{\frac{p-1}{2}}\lfloor it/p\rfloor$ and $p\langle it/p\rangle = r_i$. Among the integers in the set $\{r_1,\dots,r_{\frac{p-1}{2}}\}$, let $r_{i_1},\dots,r_{i_\mu}$ be the ones that are greater than $\frac{p-1}{2}$. Then Gauss' lemma states that $(t/p) = (-1)^\mu$. With Q as above, it is straightforward to show that $\mu \equiv (t-1)\frac{p^2-1}{8} + Q \mod 2$. Hence $(t/p) = (-1)^{(t-1)\frac{p^2-1}{8}+Q}$. Therefore, from Gauss's lemma,

$$Q = n-(t-1)\frac{p^2-1}{8}, \text{ where } n \text{ is even if } (t/p)=1, \text{ and } n \text{ is odd if } (t/p)=-1. \tag{40}$$

Therefore, from the above we have

$$\begin{aligned} p\sum_{i=1}^{\frac{p-1}{2}}(i/p-\langle ti/p\rangle) = \sum_{i=1}^{\frac{p-1}{2}}(i-(it-p\lfloor it/p\rfloor)) &= -(t-1)\frac{p^2-1}{8}+pQ\\ &= -(t-1)\frac{p^2-1}{8}+p(n-(t-1)\frac{p^2-1}{8}) \quad \text{by (40)}\\ &= -(p+1)(t-1)\frac{p^2-1}{8}+pn. \end{aligned}$$

But $p+1$ is even, and by (40) n is even (respectively odd) if $(t/p)=1$ (respectively -1). This proves the proposition. □

Proof of Theorem 16, First Case (*continued*). It follows directly from (36), (37), (39), and Proposition 17 that

$$\begin{aligned} \frac{v^f}{\sigma v^f} = 1 &\Leftrightarrow \left(\frac{v}{v_t}\right)^{f/2} \in \mathbb{Q}(\zeta_f),\\ \frac{v^f}{\sigma v^f} = -1 &\Leftrightarrow i\left(\frac{v}{v_t}\right)^{f/2} \in \mathbb{Q}(\zeta_f). \end{aligned}$$

This proves that the equivalent criterion of Theorem 17 holds, and hence proves Theorem 16 in this case (when **a** lifts canonical basis classes indexed by two distinct odd primes). □

Proof of Theorem 16, Second Case. We now turn to the case when $\mathbf{a}$ represents any of the canonical basis classes indexed by four or more distinct odd primes. Here, our proof is an easy consequence of Theorem 15.

By Proposition 16, without loss of generality, we may assume that C is a canonically lifted cycle. From Theorem 10, $v = 1$, and $\Gamma(\mathbf{a})^2 = \sin \mathbf{a}$. From Theorem 15 we conclude that $(v/v_t)^{f/2} = (1/v_t)^{f/2} \in \mathbb{Q}$. Also since $v = 1$, therefore $v^f/\sigma v^f = 1$. This shows that the equivalent criterion of Theorem 17 holds, and hence proves Theorem 16 in this case (when $\mathbf{a}$ lifts canonical basis classes indexed by at least four distinct odd primes). ☐

Let $a = \sum m_i [a_i] \in H^2(\pm, \mathbb{U})$, let f be the lcm of the denominators of the $\langle a_i \rangle$, and let f be odd. At the beginning of this section, we observed that each $\mathbf{a} \in H^2(\pm, \mathbb{U})$ gives rise to a double covering of $\mathbb{Q}(\zeta_{4f})/\mathbb{Q}$ by $\mathbb{Q}(\zeta_{4f}, \sqrt{\sin \mathbf{a}})$. Using Theorem 16, we can now prove the following stronger result:

Theorem 18. *Let $a = \sum m_i [a_i] \in H^2(\pm, \mathbb{U})$, let f be the lcm of the denominators of the $\langle a_i \rangle$, and let f be odd. Then the following are true:*

(**A**) *Let $\mathbf{a}$ represent any of the basis classes of $H^2(\pm, \mathbb{U})$, indexed by odd square-free positive integers divisible by exactly two primes. Then, for all $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$,*

$$\frac{\sqrt{\sin \mathbf{a}}}{\sigma\sqrt{\sin \mathbf{a}}} \in \mathbb{Q}(\zeta_{4f}).$$

Thus, in this case $\mathbf{a}$ gives rise to a double covering of $\mathbb{Q}(\zeta_{4f})/\mathbb{Q}$ by $\mathbb{Q}(\zeta_{4f}, \sqrt{\sin \mathbf{a}})$.

(**B**) *Let $\mathbf{a}$ represents any of the basis classes of $H^2(\pm, \mathbb{U})$, indexed by odd square-free positive integers divisible by at least four primes. Let C be a cycle in $\mathbb{SK}$ such that $C = \bigoplus_{i+j=o} C_{i,j}$, $C_{0,0} = \sum m_i [a_i, 1, 0]$ and $(\partial + \delta)C = 0$. Assume that no term of the form $[0, pq, 0]$ appears in $C_{2,-2}$. Then, for all $\sigma \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$,*

$$\frac{\sqrt{\sin \mathbf{a}}}{\sigma\sqrt{\sin \mathbf{a}}} \in \mathbb{Q}(\zeta_f).$$

Thus, in this case $\mathbf{a}$ gives rise to a double covering of $\mathbb{Q}(\zeta_f)/\mathbb{Q}$ by $\mathbb{Q}(\zeta_f, \sqrt{\sin \mathbf{a}})$.

Proof. As earlier, we may assume that $a_i \not\equiv 0 \mod \mathbb{Z}$.

(**A**) This is a consequence of Theorems 11 and 12, as observed earlier. We provide another proof, using Theorem 16. From Theorem 15, in this case we have

$$\left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right) = \omega \left(\frac{\sqrt{r}}{\sigma\sqrt{r}}\right)^{1/2f} \left(\frac{\sqrt{\sin \mathbf{a}}}{\sigma\sqrt{\sin \mathbf{a}}}\right),$$

where ω is a primitive $2f$-th root of unity, and $r \in \mathbb{Q}$. By Theorem 16, $\Gamma(\mathbf{a})/\sigma\Gamma(\mathbf{a}) \in \mathbb{Q}(\zeta_f)$. Also, since $\sqrt{r}/\sigma\sqrt{r} = \pm 1$, therefore $(\sqrt{r}/\sigma\sqrt{r})^{1/2f} \in \mathbb{Q}(\zeta_{4f})$. This proves (**A**).

(**B**) From Theorem 15, in this case we have $\Gamma(\mathbf{a})^{2f} = s(\sin \mathbf{a})^f$, where $s \in \mathbb{Q}$. Therefore

$$\left(\frac{\Gamma(\mathbf{a})}{\sigma\Gamma(\mathbf{a})}\right) = \omega \left(\frac{\sqrt{\sin \mathbf{a}}}{\sigma\sqrt{\sin \mathbf{a}}}\right),$$

where ω is a primitive $2f$-th root of unity. The proof of (**B**) now follows from Theorem 16. ☐

16. The Galois group of the extension $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a})^f)/\mathbb{Q}$

Here we want to examine the Galois group of the extension $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a})^f)/\mathbb{Q}$. Our results, once again, depend on the basis classes of $H^2(\pm, \mathbb{U})$, represented by the element $\mathbf{a} \in H^2(\pm, \mathbb{U})$. When $\mathbf{a}$ represents a basis class of $H^2(\pm, \mathbb{U})$, indexed by two odd primes, we will demonstrate by an example that the Galois group in question can be non-abelian. However, if $\mathbf{a}$ represents any of the canonical basis classes of $H^2(\pm, \mathbb{U})$, indexed by an odd squarefree positive integer divisible by at least four primes, then this Galois group is abelian. Our proof requires the following result of Deligne [3]:

Theorem 19. *Let $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$, and assume that $\sum m_i \langle a_i \rangle$ is an integer. Define*

$$\tilde{\Gamma}(\mathbf{a}) = \frac{1}{(2\pi i)^{\sum m_i \langle a_i \rangle}} \prod_{i:\ a_i \not\equiv 0} (\Gamma(\langle a_i \rangle))^{m_i}.$$

Let $\sigma, \tau \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, and let σ_t be the restriction of σ to $\mathbb{Q}(\zeta_f)$. Then

$$\sigma\left(\frac{\tilde{\Gamma}(\mathbf{a})}{\tau\tilde{\Gamma}(\mathbf{a})}\right) = \frac{\tilde{\Gamma}(\mathbf{a}^t)}{\tau\tilde{\Gamma}(\mathbf{a}^t)}.$$

Proof. For a proof, we refer to [3], Theorem 7.18(b), page 95. □

Remark. Note that the definition of the gamma monomial $\tilde{\Gamma}(\mathbf{a})$ in Deligne's theorem differs from our definition of $\Gamma(\mathbf{a})$. However, if the lcm of the denominators of the a_i is odd, then this distinction is of no consequence. In fact we have the following:

Theorem 20. *Let $\mathbf{a} = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$. Let f be the lcm of the denominators of the $\langle a_i \rangle$, and assume that f is odd. Let $\sigma, \tau \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, and let σ_t be the restriction of σ to $\mathbb{Q}(\zeta_f)$. Then*

$$\sigma\left(\frac{\Gamma(\mathbf{a})}{\tau\Gamma(\mathbf{a})}\right) = \frac{\Gamma(\mathbf{a}^t)}{\tau\Gamma(\mathbf{a}^t)}.$$

Proof. As before, we may assume that $a_i \not\equiv 0 \mod \mathbb{Z}$. Let $w = \sum m_i \langle a_i \rangle$. Directly from the definitions, and using $\sum m_i \langle a_i \rangle = \sum m_i \langle t a_i \rangle$, we have

$$\Gamma(\mathbf{a})\tilde{\Gamma}(\mathbf{a}) = \frac{(2\pi)^{\frac{\sum m_i}{2}}}{(2\pi)^{\sum m_i \langle a_i \rangle}(i)^{\sum m_i \langle a_i \rangle}}.$$

By Proposition 3, $\sum m_i \langle a_i \rangle = \frac{\sum m_i}{2}$. Therefore

$$\Gamma(\mathbf{a})\tilde{\Gamma}(\mathbf{a}) = \frac{1}{i^w}. \tag{41}$$

Since f is odd, we get from Proposition 3 $\sum m_i \equiv 0 \mod 2$, and hence w is an integer. Using (41), we have from Theorem 19

$$\sigma\left(\frac{i^w}{\tau(i^w)}\right)\sigma\left(\frac{\Gamma(\mathbf{a})}{\tau\Gamma(\mathbf{a})}\right) = \left(\frac{i^w}{\tau(i^w)}\right)\frac{\Gamma(\mathbf{a}^t)}{\tau\Gamma(\mathbf{a}^t)}.$$

But since w is an integer, therefore $i^w/\tau(i^w) = \pm 1$. Therefore

$$\sigma\left(\frac{i^w}{\tau(i^w)}\right) = \frac{i^w}{\tau(i^w)}.$$

This proves the theorem. □

First, we use Theorem 20 to discuss the case when $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a})^f)/\mathbb{Q}$ is an abelian extension. We have the following:

Theorem 21. *Let $a = \sum m_i\,[a_i] \in H^2(\pm, \mathbb{U})$, let f be the lcm of the denominators of the $\langle a_i \rangle$, and let f be odd. Assume that $\mathbf{a}$ represents any of the canonical basis classes of $H^2(\pm, \mathbb{U})$, indexed by odd squarefree positive integers divisible by at least four primes. Then $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a})^f)$ is an abelian extension of $\mathbb{Q}$.*

To prove Theorem 21, we need the following reduction:

Proposition 18. *Let $k_g \in \mathbb{SK}_{i,-i}/\mathbb{NSK}_{i,-i}$ be a canonical basis class of $H^2(\pm, \mathbb{U})$, indexed by a squarefree odd positive integer g divisible by i primes, where i is even. Let C and C' be two different cycles lifting k_g—that is, C and C' both represent the same canonical basis class of $H^2(\pm, \mathbb{U})$. Let $\mathbf{a}$ and $\mathbf{a}'$ be obtained from the bidegree $(0,0)$ components of C and C' respectively, and let $\mathbf{a} = \sum m_i a_i$, $\mathbf{a}' = \sum m_i a_i'$. Let f be the lcm of the denominators of the a_i and a_i', and assume that f is odd. Then $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a})^f)/\mathbb{Q}$ is an abelian extension if and only if the same is true for the extension $\mathbb{Q}(\zeta_f, \Gamma(\mathbf{a}')^f)/\mathbb{Q}$.*

Proof. There exists a canonically lifted chain B in $\mathbb{SK}$ such that $(\partial + \delta)B = C - C'$. As before, it follows that

$$\left(\frac{\Gamma(\mathbf{a})}{\Gamma(\mathbf{a}')}\right)^f = \sqrt{r}(\sin \mathbf{b})^f,$$

where r is a rational number and $\mathbf{b}$ is identified with $B_{0,1}$. Note that $\sqrt{r} = \Gamma(\partial B_{1,0})^f$. Clearly $\sqrt{r}(\sin \mathbf{b})^f \in \mathbb{Q}(\zeta_{4f})$. Therefore $\Gamma(\mathbf{a})^f = \eta\Gamma(\mathbf{a}')^f$, where, $\eta \in \mathbb{Q}(\zeta_{4f})$. Let $\sigma, \tau \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$. Since $\sigma\tau(\eta) = \tau\sigma(\eta)$, therefore $\sigma\tau\Gamma(\mathbf{a})^f = \tau\sigma\Gamma(\mathbf{a})^f$ if and only if $\sigma\tau\Gamma(\mathbf{a}')^f = \tau\sigma\Gamma(\mathbf{a}')^f$. This proves the proposition. □

Proof of Theorem 21. Let $\sigma, \tau \in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$, and let σ_t be the restriction of σ to $\mathbb{Q}(\zeta_f)$. By Proposition 18, without loss of generality we may assume that $\mathbf{a}$ forms the bidegree $(0,0)$ component of a canonically lifted cycle. Since $\mathbf{a}$ is obtained by canonically lifting a basis class indexed by an odd squarefree positive integer divisible by at least four primes, by Theorem 15 we have $\Gamma(\mathbf{a})^{2f} = (\sin \mathbf{a})^f$ and $\Gamma(\mathbf{a}^t)^{2f} = r^2(\sin \mathbf{a}^t)^f$, where $r \in \mathbb{Q}$. We therefore have

$$\sigma\Gamma(\mathbf{a})^{2f} = (\sigma_t \sin \mathbf{a})^f = (\sin \mathbf{a}^t)^f = (\text{by Thm. 12})\ \frac{\Gamma(\mathbf{a}^t)^{2f}}{r^2}.$$

Hence,

$$\sigma\Gamma(\mathbf{a})^f = \epsilon\frac{\Gamma(\mathbf{a}^t)^f}{r}, \quad \text{where } \epsilon = \pm 1. \tag{42}$$

Now from Theorem 20

$$\frac{\sigma\Gamma(\mathbf{a})^f}{\sigma\tau\Gamma(\mathbf{a})^f} = \frac{\Gamma(\mathbf{a}^t)^f}{\tau\Gamma(\mathbf{a}^t)^f}.$$

Substituting from (42), we get

$$\frac{\epsilon\Gamma(\mathbf{a}^t)^f}{r\sigma\tau\Gamma(\mathbf{a})^f} = \frac{\Gamma(\mathbf{a}^t)^f}{\tau(\epsilon r\sigma\Gamma(\mathbf{a})^f} = \frac{\Gamma(\mathbf{a}^t)^f}{\epsilon r\tau\sigma\Gamma(\mathbf{a})^f}.$$

Since $\epsilon^2 = 1$, we have $\sigma\tau\Gamma(\mathbf{a})^f = \tau\sigma\Gamma(\mathbf{a})^f$. This proves Theorem 21. □

Remark. We know from Theorem 15 that $\Gamma(\mathbf{a})^f = s^{1/2}(\sin\mathbf{a})^{\frac{f-1}{2}}\sqrt{\sin\mathbf{a}}$, where $s\in\mathbb{Q}$. Hence, it follows from Theorem 21 that, for all $\sigma,\tau\in Gal(\bar{\mathbb{Q}}/\mathbb{Q})$,

$$\sigma\tau\sqrt{\sin\mathbf{a}} = \tau\sigma\sqrt{\sin\mathbf{a}}.$$

By Theorem 18, $\mathbb{Q}(\zeta_f,\sqrt{\sin\mathbf{a}})$ is a Galois extension of $\mathbb{Q}$. Therefore Theorem 21 implies the following:

Theorem 22. *Let $a=\sum m_i\,[a_i]\in H^2(\pm,\mathbb{U})$, let f be the lcm of the denominators of the $\langle a_i\rangle$, and let f be odd. Assume that $\mathbf{a}$ represents any of the canonical basis classes of $H^2(\pm,\mathbb{U})$, indexed by odd squarefree positive integers divisible by at least four primes. Then $\mathbb{Q}(\zeta_f,\sqrt{\sin\mathbf{a}})$ is an abelian extension of $\mathbb{Q}$. Therefore $\sqrt{\sin\mathbf{a}}\in\mathbb{Q}(\zeta_\infty)$.*

Proof. The first part follows from the remark above. The last assertion is a consequence of the Kronecker-Weber theorem. □

Example of a non-abelian Galois extension. From Theorem 18, we know that if $\mathbf{a}$ represents any of the basis classes of $H^2(\pm,\mathbb{U})$, indexed by odd squarefree positive integers divisible by exactly two primes, then $\mathbb{Q}(\zeta_{4f},\sqrt{\sin\mathbf{a}})$ is a Galois extension of $\mathbb{Q}$. We want to show by an example that in this case the Galois group of the extension $\mathbb{Q}(\zeta_{4f},\sqrt{\sin\mathbf{a}})/\mathbb{Q}$ is in general non-abelian. We turn once again to our example of the canonically lifted cycle C_{15}, lifting the basis class $k_{15}=[0,15,-2]\in\mathbb{SK}_{2,-2}/\mathbb{NSK}_{2,-2}$ of $H^2(\pm,\mathbb{U})$. From Section 9, we have $\mathbf{a}_{15}=[1/3]-[4/15]-[1/5]+[2/15]$, representing the basis class $[0,15,-2]$ of $H^2(\pm,\mathbb{U})$, and $\Gamma(\mathbf{a}_{15})^2 = 3^{-2/5}5^{1/6}\sin\mathbf{a}_{15}$. To compute the Galois group of $\mathbb{Q}(\zeta_{60},\sqrt{\sin\mathbf{a}})$ over $\mathbb{Q}$, we use the following observation, which we state without proof:

Proposition 19. *Let $k\subset K$ be a Galois extension, and let $u\in K, u\neq 0$, be such that $K(\sqrt{u})/k$ is Galois. Assume that $\forall\sigma\in Gal(K/k)$, $\frac{u}{\sigma u}\in(K^\times)^2$. Write $G=Gal(K/k)$ and $\tilde{G}=Gal(K(\sqrt{u})/k)$. Let $Gal(K(\sqrt{u})/K)=\{1,\epsilon\}$, where $\epsilon\sqrt{u}=-\sqrt{u}$. For all $\sigma\in G$, choose $v_\sigma\in K^\times$ such that $u=v_\sigma^2(\sigma u)$. For all $\sigma\in G$, define a lifting $\tilde\sigma\in\tilde G$ by $v_\sigma\tilde\sigma\sqrt{u}=\sqrt{u}$. Finally, for all $\sigma,\tau\in G$, define $i(\sigma,\tau)\in\{0,1\}$ by $(-1)^{i(\sigma,\tau)}=v_\sigma(\sigma v_\tau)/v_{\sigma\tau}$. Then $\tilde G=\{\tilde\sigma\epsilon^j : \sigma\in G, j=0,1\}$, where the multiplication in $\tilde G$ is given by $\widetilde{\sigma\tau}=\tilde\sigma\tilde\tau\epsilon^{i(\sigma,\tau)}$.*

For our example we substitute $k=\mathbb{Q}$, $K=\mathbb{Q}(\zeta_{60})$, and $u=\sin\mathbf{a}$. Recall that, by Theorems 11, and 12, for all $\sigma_t\in Gal(\mathbb{Q}(\zeta_{60})/\mathbb{Q})$,

$$\frac{u}{\sigma_t u} = \frac{\sin\mathbf{a}}{\sin\mathbf{a}^t} = (\sin\mathbf{b})^2,$$

where $\sin\mathbf{b}\in\mathbb{Q}(\zeta_{60})$. Thus the hypothesis of Proposition 19 is satisfied. We have $Gal(\mathbb{Q}(\zeta_{60})/\mathbb{Q})\simeq\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$. We choose the generators σ_{37} (of order 4), and σ_{31},σ_{41} (both of order 2) for $Gal(\mathbb{Q}(\zeta_{60})/\mathbb{Q})$. Then, from Proposition 19, $Gal(\mathbb{Q}(\zeta_{60},\sqrt{\sin\mathbf{a}})/\mathbb{Q})$ is generated by $\{\tilde\sigma_{31},\tilde\sigma_{37},\tilde\sigma_{41},\tilde\sigma_{31}\epsilon,\tilde\sigma_{37}\epsilon,\tilde\sigma_{41}\epsilon\}$, where $\epsilon\in Gal(\mathbb{Q}(\zeta_{60},\sqrt{\sin\mathbf{a}})/\mathbb{Q}(\zeta_{60}))$, so that $\epsilon\sqrt{\sin\mathbf{a}}=-\sin\mathbf{a}$. To determine the group multiplication in $Gal(\mathbb{Q}(\zeta_{60},\sqrt{\sin\mathbf{a}})/\mathbb{Q})$, we need to compute $\{i(\sigma_s,\sigma_t) : s,t = 31,37,41\}$. For this we first need $v_{\sigma_{31}}$, $v_{\sigma_{37}}$, and $v_{\sigma_{41}}$. We note that $v_{\sigma_t}^2=\frac{\sin\mathbf{a}}{\sin\mathbf{a}^t}$. We proceed exactly as in the example following Theorem 12, and use canonical lifting to find chains B, B' and B'' such that

$$(\partial+\delta)B = C - C^{31},\quad (\partial+\delta)B' = C - C^{37},\quad (\partial+\delta)B'' = C - C^{41}.$$

Identifying $\mathbf{b}$, $\mathbf{b}'$, and $\mathbf{b}'' \in \mathbb{A}$, with $B_{0,1}$, $B'_{0,1}$, and $B''_{0,1}$, respectively, we have $v_{\sigma_{31}} = \sin \mathbf{b}$, $v_{\sigma_{37}} = \sin \mathbf{b}'$, $v_{\sigma_{41}} = \sin \mathbf{b}''$. With these computations we finally get

$$v_{\sigma_{31}} = 1, \quad v_{\sigma_{37}} = \frac{\sin(2\pi/15)}{2\sin(\pi/15)\sin(4\pi/15)}, \quad v_{\sigma_{41}} = \frac{1}{4\sin(4\pi/15)\sin(7\pi/15)}.$$

Now, using the definition of $i(\sigma_s, \sigma_t)$, we obtain

$$\begin{array}{lll} i(\sigma_{31},\sigma_{31}) = 0, & i(\sigma_{31},\sigma_{37}) = 0, & i(\sigma_{31},\sigma_{41}) = 0, \\ i(\sigma_{37},\sigma_{31}) = 0, & i(\sigma_{37},\sigma_{37}) = 1, & i(\sigma_{37},\sigma_{41}) = 0, \\ i(\sigma_{41},\sigma_{31}) = 0, & i(\sigma_{41},\sigma_{37}) = 1, & i(\sigma_{41},\sigma_{41}) = 1. \end{array}$$

Using the definition of group multiplication in $Gal(\mathbb{Q}(\zeta_{60}, \sqrt{\sin \mathbf{a}})/\mathbb{Q})$, we determine the generators and relations. Thus

$$Gal(\mathbb{Q}(\zeta_{60}, \sqrt{\sin \mathbf{a}})/\mathbb{Q}) = \langle \{\tilde{\sigma}_{31}, \tilde{\sigma}_{37}, \tilde{\sigma}_{41}, \tilde{\sigma}_{31}\epsilon, \tilde{\sigma}_{37}\epsilon, \tilde{\sigma}_{41}\epsilon\} \rangle,$$

where $\epsilon \in Gal(\mathbb{Q}(\zeta_{60}, \sqrt{\sin \mathbf{a}})/\mathbb{Q}(\zeta_{60}))$ such that $\epsilon\sqrt{\sin \mathbf{a}} = -\sqrt{\sin \mathbf{a}}$. The generators described above satisfy the relations

$$\epsilon^2 = \tilde{\sigma}_{31}^2 = \tilde{\sigma}_{37}^4 = \tilde{\sigma}_{41}^4 = 1,$$
$$\tilde{\sigma}_{31}^{-1}\tilde{\sigma}_{37}^{-1}\tilde{\sigma}_{31}\tilde{\sigma}_{37} = \tilde{\sigma}_{31}^{-1}\tilde{\sigma}_{41}^{-1}\tilde{\sigma}_{31}\tilde{\sigma}_{41} = 1, \ \tilde{\sigma}_{37}^{-1}\tilde{\sigma}_{41}^{-1}\tilde{\sigma}_{37}\tilde{\sigma}_{41} = \epsilon.$$

Observe that the final relation listed above shows that $Gal(\mathbb{Q}(\zeta_{60}, \sqrt{\sin \mathbf{a}})/\mathbb{Q})$ is non-abelian.

17. Conclusion

We conclude our discussions by indicating some possible directions for further research.

Unit index formula. From Theorem 22, we know that if $\mathbf{a}$ represents any of the canonical basis classes of $H^2(\pm, \mathbb{U})$, indexed by odd squarefree positive integers divisible by at least four primes, then $\sqrt{\sin \mathbf{a}} \in \mathbb{Q}(\zeta_\infty)$. The $\sqrt{\sin \mathbf{a}}$ may very well be a new supply of abelian units. Given $\mathbf{a}$ as above, it is important to determine the smallest cyclotomic extension of $\mathbb{Q}$ which contains the unit $\sqrt{\sin \mathbf{a}}$. We want to investigate the relevance of these units to the unit index formula for cyclotomic fields. For relevant results on cyclotomic units, we recommend the texts by Washington [9] and by Lang [7]. Let $\mathbb{Q}(\zeta_n)^+$ be the maximal real subfield of $\mathbb{Q}(\zeta_n)$, and let E_n^+ be the unit group of $\mathbb{Q}(\zeta_n)^+$. Furthermore let C_n^+ be the cyclotomic units of $\mathbb{Q}(\zeta_n)^+$, and let h_n^+ be the class number of $\mathbb{Q}(\zeta_n)^+$. Sinnott [8] has computed the index for the group of cyclotomic units of $\mathbb{Q}(\zeta_n)^+$ in the full group of units E_n^+. Sinnott's unit index formula gives

$$[E_n^+ : C_n^+] = 2^b h_n^+, \tag{43}$$

where $g =$ number of prime factors of n, $b = 0$ if $g = 1$, and $b = 2^{g-2} + 1 - g$ if $g \geq 2$. There has been much interest in finding an improved version of the unit index formula (43) that does not involve the "parasitic factors" of 2. Perhaps it is possible to enlarge the group of cyclotomic units C_n^+ by adjoining the real units of the form $\sqrt{\sin \mathbf{a}}$, and kill the factors of 2 in formula (43). As mentioned earlier, the first step in this direction is to determine the smallest cyclotomic extension of $\mathbb{Q}$ which contains the unit $\sqrt{\sin \mathbf{a}}$.

Computing the Galois groups of $\mathbb{Q}(\zeta_f, \sqrt{\sin \mathbf{a}}/\mathbb{Q})$. In Section 16 we outlined a procedure for computing the Galois group of the extension $\mathbb{Q}(\zeta_f, \sqrt{\sin \mathbf{a}})$ over $\mathbb{Q}$. We hope to be able to provide a complete description of the Galois groups in question.

The case when $2 \mid f$. Here, for simplicity, we restrict ourselves to $\mathbf{a} = \sum m_i \langle a_i \rangle \in H^2(\pm, \mathbb{U})$ for which f is odd, where f is the lcm of the denominators of the $\langle a_i \rangle$.We believe that most of our results can be generalised to the case when $2 \mid f$. This can be achieved by fine tuning our methods and making appropriate use of the duplication formula for gamma functions. For example, by using a method analogous to canonical lifting, we can lift the basis class

$$k_{2p} = [0, 2p, -2] + [1/2, 2p, -2] \in \mathbb{SK}_{2,-2}/\mathbb{NSK}_{2,-2},$$

of $H^2(\pm, \mathbb{U})$, and read off $\mathbf{a}_{2p}$ from the $(0,0)$ component of the lifted cycle. In this case we can show that

$$\Gamma(\mathbf{a}_{2p})^2 = \frac{p^{1/4}}{2^{(p-1)/4}} \sin \mathbf{a}_{2p}.$$

In particular, for $p = 3$ we have $\mathbf{a}_6 = [1/4] - [1/3] - [5/12]$, and therefore

$$\frac{1}{2\pi}\left(\frac{\Gamma(1/3)\Gamma(5/12)}{\Gamma(1/4)}\right)^2 = \frac{3^{1/4}}{2^{1/2}}\left(\frac{\sin(\pi/4)}{2\sin(\pi/3)\sin(5\pi/12)}\right).$$

Sufficient criterion for H^1. It should be possible to show that our (necessary) criterion for an element of $\mathbb{A}$ to be in $H^1(\pm, \mathbb{U})$ is also sufficient. The proof should be quite similar to the proof of the Koblitz-Ogus criterion [4], which uses values of the L-function at $s = 0$. Here too, we need to treat the case when $2 \mid f$ separately.

Elliptic and modular generalisations. Elliptic and modular generalisations of our methods need to be investigated. Of particular interest is to determine if an analogous theory of double coverings provides a method for generating elliptic and modular units.

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References

1. G. W. Anderson, *A double complex for computing the sign-cohomology of the universal ordinary distribution*, Recent Progress in Algebra (Taejon/Seoul, 1997; S. G. Hahn et al., eds.), Contemp. Math., vol. 224, Amer. Math. Soc., Providence, RI, 1999, pp. 1–27. MR **99k:**11169
2. P. Deligne, *Valeurs de fonctions L et périodes d'intégrales*, Proc. Sympos. Pure Math. **33** (1979), part 2, 313-346. MR **81d:**12009
3. P. Deligne, J. S. Milne, A. Ogus, K.-Y. Shih, *Hodge Cycles, Motives and Shimura Varieties*, Lecture Notes in Math., vol. **900**, Springer-Verlag, New York, 1982. MR **84m:**14046
4. N. Koblitz, A. Ogus, *Algebraicity of some products of values of the* Γ *function, Appendix: Valeurs de fonctions L et périodes d'intégrales*, Proc. Sympos. Pure Math. **33** (1979), part 2, 343-346. MR **81d:**12009
5. D. S. Kubert, *The universal ordinary distribution*, Bull. Soc. Math. France, **107** (1979), 179-202. MR **81b:**12004
6. D. S. Kubert, *The* $\mathbb{Z}/2\mathbb{Z}$*- cohomology of the universal ordinary distribution*, Bull. Soc. Math. France, **107** (1979), 203-224. MR **81a:**20062

7. S. Lang, *Cyclotomic Fields, I and II*, Graduate Texts in Mathematics, **121**, 1990, Springer-Verlag, New York. MR **91c:**11001
8. W. Sinnott, *On the Stickelberger ideal and the circular units of a cyclotomic field*, Ann. Math., **108** (1978), 107-134. MR **82i:**12004
9. L. W. Washington, *Introduction to Cyclotomic Fields*, Graduate Texts in Mathematics, **83**, 1996, Springer-Verlag, New York. MR **97h:**11130

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